

# Soliton Theory, Symmetric Functions and Matrix Integrals

A. Yu. Orlov\*

## Abstract

We consider certain scalar product of symmetric functions which is parameterized by a function  $r$  and an integer  $n$ . On the one hand we have a fermionic representation of this scalar product. On the other hand we get a representation of this product with the help of multi-integrals. This gives links between a theory of symmetric functions, soliton theory and models of random matrices (such as a model of normal matrices).

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\*Oceanology Institute, Nahimovskii prospect 36, Moscow, Russia, email address: orlovs@wave.sio.rssi.ru

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# 1 Introduction

Our goal is to make links between soliton theory, theory of symmetric functions and random matrices. We shall consider a certain class of tau-functions which we call tau-functions of hypergeometric type. The key point of the paper is to identify scalar products on the space of symmetric functions with vacuum expectations of fermionic fields.

We connect different topics: integrals over matrices, hypergeometric functions, theory of symmetric functions. We also show that a scalar product of tau-functions is also a tau-function. The paper is partially based on [9] and [5].

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**Symmetric functions.** Symmetric function is a function of variables  $\mathbf{x} = x_1, \dots, x_n$  which is invariant under the action of permutation group  $S_n$ . Symmetric polynomials of  $n$  variables form a ring denoted by  $\Lambda_n$ . The number  $n$  is usually irrelevant and may be infinite. The ring of symmetric functions in infinitely many variables is denoted by  $\Lambda$ .

Power sums are defined as

$$p_m = \sum_{i=1} x_i^m, \quad m = 1, 2, \dots \quad (1.0.1)$$

Functions  $p_{m_1} p_{m_2} \cdots p_{m_k}$ ,  $k = 1, 2, \dots$  form a basis in  $\Lambda$ .

There are different well investigated polynomials of many variables such as Schur functions, projective Schur functions, complete symmetric functions [1]. Each set form a basis in  $\Lambda$ .

There exists a scalar product in  $\Lambda$  such that Schur functions are orthonormal functions:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \quad (1.0.2)$$

The notion of scalar product is very important in the theory of symmetric functions. The choice of scalar product gives different orthogonal and bi-orthogonal systems of polynomials.

**Soliton theory.** KP hierarchy of integrable equations, which is the most popular example, consists of semi-infinite set of nonlinear partial differential equations

$$\partial_{t_m} u = K_m[u], \quad m = 1, 2, \dots \quad (1.0.3)$$

which are commuting flows:

$$[\partial_{t_k}, \partial_{t_m}] u = 0 \quad (1.0.4)$$

The first nontrivial one is Kadomtsev-Petviashvili equation

$$\partial_{t_3} u = \frac{1}{4} \partial_{t_1}^3 u + \frac{3}{4} \partial_{t_1}^{-1} \partial_{t_2}^2 u + \frac{3}{4} \partial_{t_1} u^2 \quad (1.0.5)$$

which originally served in plasma physics [13]. This equation, which was integrated in [17],[3], now plays a very important role both, in physics (see [14]; see review in [15] for modern applications) and in mathematics. Each members of the KP hierarchy is compatible with each other one.

Another very important equation is the equation of two-dimensional Toda lattice (TL) [18],[4],[23]

$$\partial_{t_1} \partial_{t_1^*} \phi_n = e^{\phi_{n-1} - \phi_n} - e^{\phi_n - \phi_{n+1}} \quad (1.0.6)$$

This equation gives rise to TL hierarchy which contains derivatives with respect to higher times  $t_1, t_2, \dots$  and  $t_1^*, t_2^*, \dots$

The key point of soliton theory is the notion of tau function, introduced by Sato (for KP tau-function see [4]). Tau function is a sort of potential which gives rise both to TL hierarchy and KP hierarchy. It depends on two semi-infinite sets of higher times  $t_1, t_2, \dots$  and  $t_1^*, t_2^*, \dots$ , and discrete variable  $n$ :  $\tau = \tau(n, \mathbf{t}, \mathbf{t}^*)$ . More explicitly we have [4],[19, 16],[23]:

$$u = 2\partial_{t_1}^2 \log \tau(n, \mathbf{t}, \mathbf{t}^*), \quad \phi_n(\mathbf{t}, \mathbf{t}^*) = -\log \frac{\tau(n+1, \mathbf{t}, \mathbf{t}^*)}{\tau(n, \mathbf{t}, \mathbf{t}^*)} \quad (1.0.7)$$

In soliton theory so-called Hirota-Miwa variables  $\mathbf{x}, \mathbf{y}$ , which are related to higher times as

$$mt_m = \sum_i x_i^m, \quad mt_m^* = \sum_i y_i^m, \quad (1.0.8)$$

are well-known [20]. Tau function is a symmetric function in Hirota-Miwa variables, higher times  $t_m, t_m^*$  are basically power sums, see (1.0.1), see [37] as an example.

It is known fact that typical tau function may be presented in the form of double series in Schur functions [24, 25, 26]:

$$\tau(n, \mathbf{t}, \mathbf{t}^*) = \sum_{\lambda, \mu} K_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\mathbf{t}^*) \quad (1.0.9)$$

where coefficients  $K_{\lambda\mu}$  are independent of higher times and solve very special bilinear equations, equivalent to Hirota bilinear equations.

In the previous paper we use these equations to construct hypergeometric functions which depend on many variables, these variables are KP and Toda lattice higher times. Here we

shall use the general approach to integrable hierarchies of Kyoto school [4], see also [19, 16]. Especially a set of papers about Toda lattice [23, 24, 25, 26, 27, 28, 29] is important for us. Let us notice that there is an interesting set of papers of M.Adler and P. van Moerbeke, where links between solitons and random matrix theory were worked out from a different point of view (see [10],[11] and other papers).

**Random matrices.** In many problems in physics and mathematics one use probability distribution on different sets of matrices. We are interested mainly in two matrix models. The integration measure  $dM_{1,2}$  is different for different ensembles

$$Z = \int e^{V_1(M_1)+V_2(M_2)+U(M_1,M_2)} dM_1 dM_2 \quad (1.0.10)$$

where

$$V_1(M_1) = Tr \sum_{m=1}^{\infty} M_1^m t_m, \quad V_2(M_2) = Tr \sum_{m=1}^{\infty} M_2^m t_m^* \quad (1.0.11)$$

while  $U$  may be different for different ensembles.

The crucial point when evaluating the integral (1.0.10) is the possibility to reduce it to the integral over eigenvalues of matrices  $M_1, M_2$ . Then (1.0.10) takes the form

$$Z = C \int e^{\sum_{i=1}^n \sum_{m=1}^{\infty} (x_i^m t_m + y_i^m t_m^*)} e^{U(\mathbf{x}, \mathbf{y})} \prod_{i=1}^n dx_i dy_i \quad (1.0.12)$$

## 2 Symmetric functions

### 2.1 Schur functions and scalar product [1]

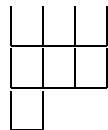
**Partitions.** Polynomial functions of many variables are parameterized by partitions. A *partition* is any (finite or infinite) sequence of non-negative integers in decreasing order:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \dots \quad (2.1.1)$$

Non-zero  $\lambda_i$  in (2.1.1) are called the *parts* of  $\lambda$ . The number of parts is the *length* of  $\lambda$ , denoted by  $l(\lambda)$ . The sum of the parts is the *weight* of  $\lambda$ , denoted by  $|\lambda|$ . If  $n = |\lambda|$  we say that  $\lambda$  is the *partition of  $n$* . The partition of zero is denoted by 0.

The *diagram* of a partition (or *Young diagram*) may be defined as the set of points (or nodes)  $(i, j) \in \mathbb{Z}^2$  such that  $1 \leq j \leq \lambda_i$ . Thus Young diagram is viewed as a subset of entries in a matrix with  $l(\lambda)$  lines and  $\lambda_1$  rows. We shall denote the diagram of  $\lambda$  by the same symbol  $\lambda$ .

For example



is the diagram of  $(3, 3, 1)$ . The weight of this partition is 7, the length is equal to 3.

The *conjugate* of a partition  $\lambda$  is the partition  $\lambda'$  whose diagram is the transpose of the diagram  $\lambda$ , i.e. the diagram obtained by reflection in the main diagonal. The partition  $(3, 2, 2)$  is conjugated to  $(3, 3, 1)$ , the corresponding diagram is





There are different notations for partitions. For instance sometimes it is convenient to indicate the number of times each integer occurs as a part:

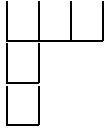
$$\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots) \quad (2.1.2)$$

means that exactly  $m_i$  of the parts of  $\lambda$  are equal to  $i$ . For instance partition  $(3, 3, 1)$  is denoted by  $(1^1 3^2)$ .

Another notation is due to Frobenius. Suppose that the main diagonal of the diagram of  $\lambda$  consists of  $r$  nodes  $(i, i)$  ( $1 \leq i \leq r$ ). Let  $\alpha_i = \lambda_i - i$  be the number of nodes in the  $i$ th row of  $\lambda$  to the right of  $(i, i)$ , for  $1 \leq i \leq r$ , and let  $\beta_i = \lambda'_i - i$  be the number of nodes in the  $i$ th column of  $\lambda$  below  $(i, i)$ , for  $1 \leq i \leq r$ . We have  $\alpha_1 > \alpha_2 > \dots > \alpha_r \geq 0$  and  $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$ . Then we denote the partition  $\lambda$  by

$$\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) = (\alpha | \beta) \quad (2.1.3)$$

One may say that Frobenius notation corresponds to hook decomposition of diagram of  $\lambda$ , where the biggest hook is  $(\alpha_1 | \beta_1)$  then  $(\alpha_2 | \beta_2)$  and so on up to the smallest one which is  $(\alpha_r | \beta_r)$ . The corners of the hooks are situated on the main diagonal of the diagram. For instance the partition  $(3, 3, 1)$  consists of two hooks  $(2, 2)$  and  $(0, 1)$ :



and



In Frobenius notation this is  $(2, 0 | 2, 1)$ .

**Schur functions, complete symmetric functions and power sums.** We consider polynomial symmetric functions of variables  $x_i, i = 1, 2, \dots, n$ , where the number  $n$  is irrelevant and may be infinity. The collection of variables  $x_1, x_2, \dots$  we denote by  $\mathbf{x}$ .

The symmetric polynomials with rational integer coefficients in  $n$  variables form a ring which is denoted by  $\Lambda_n$ . A ring of symmetric functions in countably many variables is denoted by  $\Lambda$  (a precise definition of  $\Lambda$  see in [1]).

Let us remind some of well-known symmetric functions.

For each  $m \geq 1$  the  $m$ th *power sum* is

$$p_m = \sum_i x_i^m \quad (2.1.4)$$

The collection of variables  $p_1, p_2, \dots$  we denote by  $\mathbf{p}$ .

If we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \quad (2.1.5)$$

for each partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ , then the  $p_\lambda$  form a basis of symmetric polynomial functions with rational coefficients.

Having in mind the links with soliton theory it is more convenient to consider the following "power sums":

$$\gamma_m = \frac{1}{m} \sum_i x_i^m \quad (2.1.6)$$

which is the same as Hirota-Miwa change of variables (1.0.8). The vector

$$\gamma = (\gamma_1, \gamma_2, \dots) \quad (2.1.7)$$

is an analog of the higher times variables in KP theory.

*Complete symmetric functions*  $h_n$  are given by the generating function

$$\prod_{i \geq 1} (1 - x_i z)^{-1} = \sum_{n \geq 1} h_n(\mathbf{x}) z^n \quad (2.1.8)$$

If we define  $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$  for any  $\lambda$ , then  $h_\lambda$  form a  $Z$ -basis of  $\Lambda$ .

The complete symmetric functions are expressed in terms of power sums with the help of the following generating function:

$$\exp \sum_{m=1}^{\infty} \gamma_m z^m = \sum_{n \geq 1} h_n(\gamma) z^n \quad (2.1.9)$$

Now suppose that  $n$ , the number of variables  $x_i$ , is finite.

Given partition  $\lambda$  the *Schur function*  $s_\lambda$  is the quotient

$$s_\lambda(\mathbf{x}) = \frac{\det \left( x_i^{\lambda_i + n - j} \right)_{1 \leq i, j \leq n}}{\det \left( x_i^{n - j} \right)_{1 \leq i, j \leq n}} \quad (2.1.10)$$

where denominator is the Vandermonde determinant

$$\det \left( x_i^{n - j} \right)_{1 \leq i, j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad (2.1.11)$$

Both the numerator and denominator are skew-symmetric, therefore Schur function in variables  $x_1, \dots, x_n$  is symmetric.

One puts  $s_0(\mathbf{x}) = 1$ .

The Schur functions  $s_\lambda(x_1, \dots, x_n)$ , where  $l(\lambda) \leq n$ , form a  $Z$ -basis of  $\Lambda_n$ .

Each Schur function  $s_\lambda$  can be expressed as a polynomial in the complete symmetric functions  $h_n$ :

$$s_\lambda = \det (h_{\lambda_i - i + j})_{1 \leq i, j \leq n} \quad (2.1.12)$$

where  $n \geq l(\lambda)$ .

**Orthogonality.** One may define a scalar product on  $\Lambda$  by requiring that for any pair of partitions  $\lambda$  and  $\mu$  we have

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \quad (2.1.13)$$

where  $\delta_{\lambda\mu}$  is a Kronecker symbol and

$$z_\lambda = \prod_{i \geq 1} i^{m_i} \cdot m_i! \quad (2.1.14)$$

where  $m_i = m_i(\lambda)$  is the number of parts of  $\lambda$  equal to  $i$ .

For any pair of partitions  $\lambda$  and  $\mu$  we also have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \quad (2.1.15)$$

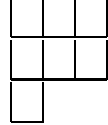
so that the  $s_\lambda$  form an *orthonormal* basis of  $\Lambda$ . The  $s_\lambda$  such that  $|\lambda| = n$  form an orthonormal basis for all symmetric polynomials of degree  $n$ .

## 2.2 A deformation of the scalar product and series of hypergeometric type

**Scalar product**  $\langle, \rangle_{r,n}$ . Let us consider a function  $r$  which depends on a single variable  $n$ , the  $n$  is integer. Given partition  $\lambda$  let us define

$$r_\lambda(x) = \prod_{i,j \in \lambda} r(x + j - i) \quad (2.2.1)$$

Thus  $r_\lambda(n)$  is a product of functions  $r$  over all nodes of Young diagram of the partition  $\lambda$  where argument of  $r$  is defined by entries  $i, j$  of a node. The value of  $j - i$  is zero on the main diagonal; the value  $j - i$  is called the content of the node. For instance, for the partition  $(3, 3, 1)$  the diagram is



so our  $r_\lambda(x)$  is equal to  $r(x+2)(r(x+1))^2(r(x))^2r(x-1)r(x-2)$ .

Given integer  $n$  and a function on the lattice  $r(n), n \in \mathbb{Z}$ , define a scalar product, parameterized by  $r, n$  as follows

$$\langle s_\lambda, s_\mu \rangle_{r,n} = r_\lambda(n) \delta_{\lambda\mu} \quad (2.2.2)$$

where  $\delta_{\lambda\mu}$  is Kronecker symbol.

Let  $n_i \in \mathbb{Z}$  are zeroes of  $r$ , and  $k = \min |n - n_i|$ . The product is non-degenerated on  $\Lambda_k$ . Really, if  $k = n - n_i > 0$  due to the definition (2.2.1) the factor  $r_\lambda(n)$  never vanish for the partitions which length is no more then  $k$ . In this case the Schur functions of  $k$  variables  $\{s_\lambda(\mathbf{x}^k), l(\lambda) \leq k\}$  form a basis on  $\Lambda_k$ . If  $n - n_i = -k < 0$  then the factor  $r_\lambda(n)$  never vanish for the partitions  $\{\lambda : l(\lambda') \leq k\}$ , where  $\lambda'$  is the conjugated partition, then  $\{s_\lambda(\mathbf{x}^k), l(\lambda') \leq k\}$  form a basis on  $\Lambda_k$ .

If  $r$  is non-vanishing function then the scalar product is non-degenerated on  $\Lambda_\infty$ .

**A useful formula.** Let  $\gamma = (\gamma_1, \gamma_2, \dots)$  and  $\gamma^* = (\gamma_1^*, \gamma_2^*, \dots)$  be two independent semi-infinite set of variables. Then we have the following useful version of Cauchy-Littlewood formula:

$$\exp \sum_{m=1}^{\infty} m \gamma_m \gamma_m^* = \sum_{\lambda} s_\lambda(\gamma) s_\lambda(\gamma^*) \quad (2.2.3)$$

where the sum is going over all partitions including zero partition and  $s_\lambda$  are constructed via (2.1.12) and (2.1.9).

**Series of hypergeometric type.** Let us consider the following pair of functions of variables  $p_m$

$$\exp \sum_{m=1}^{\infty} m \gamma_m t_m, \quad \exp \sum_{m=1}^{\infty} m \gamma_m t_m^* \quad (2.2.4)$$

where  $t_m$  and  $t_m^*$  are formal parameters. Considering  $p_m$  as power sums and using formulae (2.2.3) and (2.2.2), we evaluate the scalar product of functions (2.2.4):

$$\langle \exp \sum_{m=1}^{\infty} m \gamma_m t_m, \exp \sum_{m=1}^{\infty} m \gamma_m t_m^* \rangle_{r,n} = \sum_{\lambda} r_\lambda(n) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{t}^*) \quad (2.2.5)$$

where sum is going over all partitions including zero. The Schur functions  $s_\lambda(\mathbf{t}), s_\lambda(\mathbf{t}^*)$  are defined with the help of

$$s_\lambda(\mathbf{t}) = \det h_{\lambda_i - i + j}(\mathbf{t})_{1 \leq i, j \leq l(\lambda)}, \quad \exp \sum_{m=1}^{\infty} z^m t_m = \sum_{k=1}^{\infty} z^k h_k(\mathbf{t}) \quad (2.2.6)$$

We call the series (2.2.5) *the series of hypergeometric type*. It is not necessarily convergent series. If one chooses  $r$  as rational function he obtains so-called hypergeometric functions of matrix argument, or, in other words, hypergeometric functions related to  $GL(N, C)/U(N)$  zonal spherical functions [35],[22]. If one takes trigonometric function  $r$  we get  $q$ -deformed versions of these functions, suggested by I.G.Macdonald and studied by S.Milne in [34].

**Scalar product of series of hypergeometric type.** Considering the scalar product of functions of  $\gamma_m$  and using formulae (2.2.5) and (2.2.2), we evaluate the scalar product of series of hypergeometric type (2.2.5), and find that the answer is a series of hypergeometric type again:

$$\langle \tau_{r_1}(k, \mathbf{t}, \gamma), \tau_{r_2}(m, \gamma, \mathbf{t}^*) \rangle_{r,n} = \tau_{r_3}(0, \mathbf{t}, \mathbf{t}^*) \quad (2.2.7)$$

where

$$r_3(i) = r_1(k+i)r_2(m+i)r(n+i) \quad (2.2.8)$$

### 3 Solitons, free fermions and fermionic realization of the scalar product

In this section we use the language of free fermions as it was suggested in [19],[4].

#### 3.1 Fermionic operators, Fock spaces, vacuum expectations, tau functions [4]

**Free fermions.** We have the algebra of free fermions *the Clifford algebra*  $\mathbf{A}$  over  $\mathbf{C}$  with generators  $\psi_n, \psi_n^* (n \in \mathbb{Z})$ , satisfying the defining relations:

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0; \quad [\psi_m, \psi_n^*]_+ = \delta_{mn}. \quad (3.1.1)$$

Any element of  $W = (\oplus_{m \in \mathbb{Z}} \mathbf{C}\psi_m) \oplus (\oplus_{m \in \mathbb{Z}} \mathbf{C}\psi_m^*)$  will be referred as a *free fermion*. The Clifford algebra has a standard representation (*Fock representation*) as follows. Put  $W_{an} = (\oplus_{m < 0} \mathbf{C}\psi_m) \oplus (\oplus_{m \geq 0} \mathbf{C}\psi_m^*)$ , and  $W_{cr} = (\oplus_{m \geq 0} \mathbf{C}\psi_m) \oplus (\oplus_{m < 0} \mathbf{C}\psi_m^*)$ , and consider left (resp. right)  $\mathbf{A}$ -module  $F = \mathbf{A}/\mathbf{A}W_{an}$  (resp  $F^* = W_{cr}\mathbf{A}/\mathbf{A}$ ). These are cyclic  $\mathbf{A}$ -modules generated by the vectors  $|0\rangle = 1 \bmod \mathbf{A}W_{an}$  (resp. by  $\langle 0| = 1 \bmod W_{cr}\mathbf{A}$ ), with the properties

$$\psi_m|0\rangle = 0 \quad (m < 0), \quad \psi_m^*|0\rangle = 0 \quad (m \geq 0), \quad (3.1.2)$$

$$\langle 0|\psi_m = 0 \quad (m \geq 0), \quad \langle 0|\psi_m^* = 0 \quad (m < 0). \quad (3.1.3)$$

Vectors  $\langle 0|$  and  $|0\rangle$  are referred as left and right vacuum vectors. Fermions  $w \in W_{an}$  eliminate left vacuum vector, while fermions  $w \in W_{cr}$  eliminate right vacuum vector.

The Fock spaces  $F$  and  $F^*$  are dual ones, the pairing is defined with the help of a linear form  $\langle 0|0\rangle$  on  $\mathbf{A}$  called the *vacuum expectation value*. It is given by

$$\langle 0|1|0\rangle = 1, \quad \langle 0|\psi_m\psi_m^*|0\rangle = 1 \quad m < 0, \quad \langle 0|\psi_m^*\psi_m|0\rangle = 1 \quad m \geq 0, \quad (3.1.4)$$

$$\langle 0|\psi_m\psi_n|0\rangle = \langle 0|\psi_m^*\psi_n^*|0\rangle = 0, \quad \langle 0|\psi_m\psi_n^*|0\rangle = 0 \quad m \neq n. \quad (3.1.5)$$

and by **the Wick rule**, which is

$$\langle 0|w_1 \cdots w_{2n+1}|0\rangle = 0, \quad \langle 0|w_1 \cdots w_{2n}|0\rangle = \sum_{\sigma} \text{sgn} \sigma \langle 0|w_{\sigma(1)}w_{\sigma(2)}|0\rangle \cdots \langle 0|w_{\sigma(2n-1)}w_{\sigma(2n)}|0\rangle, \quad (3.1.6)$$



where  $w_k \in W$ , and  $\sigma$  runs over permutations such that  $\sigma(1) < \sigma(2), \dots, \sigma(2n-1) < \sigma(2n)$  and  $\sigma(1) < \sigma(3) < \dots < \sigma(2n-1)$ .

**Lie algebra  $\widehat{gl}(\infty)$ .** Consider infinite matrices  $(a_{ij})_{i,j \in \mathbb{Z}}$  satisfying the condition: there exists an  $N$  such that  $a_{ij} = 0$  for  $|i - j| > N$ ; these matrices are called generalized Jacobian matrices. The generalized Jacobian matrices form Lie algebra, Lie bracket being the usual commutator of matrices  $[A, B] = AB - BA$ .

Let us consider a set of linear combinations of quadratic elements  $\sum a_{ij} : \psi_i \psi_j^* :$ , where the notation  $::$  means *the normal ordering* which is defined as  $: \psi_i \psi_j^* := \psi_i \psi_j^* - \langle 0 | \psi_i \psi_j^* | 0 \rangle$ . These elements together with 1 span an infinite dimensional Lie algebra  $\widehat{gl}(\infty)$ :

$$[\sum a_{ij} : \psi_i \psi_j^* :, \sum b_{ij} : \psi_i \psi_j^* :] = \sum c_{ij} : \psi_i \psi_j^* : + c_0, \quad (3.1.7)$$

$$c_{ij} = \sum_k a_{ik} b_{kj} - \sum_k b_{ik} a_{kj}, \quad (3.1.8)$$

and the last term

$$c_0 = \sum_{i < 0, j \geq 0} a_{ij} b_{ji} - \sum_{i \geq 0, j < 0} a_{ij} b_{ji}. \quad (3.1.9)$$

commutes with each quadratic term.

We see that Lie algebra of quadratic elements  $\sum a_{ij} : \psi_i \psi_j^* :$  is different of the algebra of the generalized Jacobian matrices; the difference is the *central extension*  $c_0$ .

**Bilinear identity.** Now we define the operator  $g$  which is an element of the group corresponding to the Lie algebra  $\widehat{gl}(\infty)$ :

$$g = \exp \sum a_{nm} : \psi_n \psi_m : \quad (3.1.10)$$

Using (3.1.7) it is possible to derive the following relation

$$g \psi_n = \sum_m \psi_m A_{mn} g, \quad \psi_n^* g = g \sum_m A_{nm} \psi_m^*. \quad (3.1.11)$$

where coefficients  $A_{nm}$  are defined by  $a_{nm}$ . In turn (3.1.11) yields

$$[\sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^*, g \otimes g] = 0 \quad (3.1.12)$$

The last relation is very important for applications in soliton theory, and it is equivalent to the so-called Hirota equations.

**Higher times.** Let us introduce

$$H_n = \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+n}^*, \quad n \neq 0, \quad H(\mathbf{t}) = \sum_{n=1}^{+\infty} t_n H_n, \quad H^*(\mathbf{t}^*) = \sum_{n=1}^{+\infty} t_n^* H_{-n}. \quad (3.1.13)$$

$H_n \in \widehat{gl}(\infty)$ , and  $H(\mathbf{t}), H^*(\mathbf{t}^*)$  belong to  $\widehat{gl}(\infty)$  if one restricts the number of non-vanishing parameters  $t_m, t_m^*$ . For  $H_n$  we have Heisenberg algebra commutation relations:

$$[H_n, H_m] = n \delta_{m+n, 0}. \quad (3.1.14)$$

For future purposes we introduce the following fermions

$$\psi(z) = \sum_k \psi_k z^k, \quad \psi^*(z) = \sum_k \psi_k^* z^{-k-1} dz \quad (3.1.15)$$

Using (3.1.1) and (3.1.13) one obtains

$$e^{H(\mathbf{t})}\psi(z)e^{-H(\mathbf{t})} = \psi(z)e^{\xi(\mathbf{t},z)}, \quad e^{H(\mathbf{t})}\psi^*(z)e^{-H(\mathbf{t})} = \psi^*(z)e^{-\xi(\mathbf{t},z)}, \quad (3.1.16)$$

$$e^{-H^*(\mathbf{t}^*)}\psi(z)e^{H^*(\mathbf{t}^*)} = \psi(z)e^{-\xi(\mathbf{t}^*,z^{-1})}, \quad e^{-H^*(\mathbf{t}^*)}\psi^*(z)e^{H^*(\mathbf{t}^*)} = \psi^*(z)e^{\xi(\mathbf{t}^*,z^{-1})}. \quad (3.1.17)$$

Using

$$\sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^* = \text{res}_{z=0} \psi(z) \otimes \psi^*(z) \quad (3.1.18)$$

and formulae (3.1.16), (3.1.17) result in

$$\left[ \sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^*, e^{H(\mathbf{t})} \otimes e^{H(\mathbf{t})} \right] = 0, \quad \left[ \sum_{n \in \mathbb{Z}} \psi_n \otimes \psi_n^*, e^{H^*(\mathbf{t}^*)} \otimes e^{H^*(\mathbf{t}^*)} \right] = 0 \quad (3.1.19)$$

Therefore if  $g$  solves (3.1.12) then  $e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)}$  also solves (3.1.12):

$$\left[ \text{res}_{z=0} \psi(z) \otimes \psi^*(z), e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)} \otimes e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)} \right] = 0 \quad (3.1.20)$$

**The KP and TL tau functions** First let us define vacuum vectors labelled by the integer  $M$ :

$$\langle n | = \langle 0 | \Psi_n^*, \quad | n \rangle = \Psi_n | 0 \rangle, \quad (3.1.21)$$

$$\begin{aligned} \Psi_n &= \psi_{n-1} \cdots \psi_1 \psi_0 \quad n > 0, & \Psi_n &= \psi_n^* \cdots \psi_{-2}^* \psi_{-1}^* \quad n < 0, \\ \Psi_n^* &= \psi_0^* \psi_1^* \cdots \psi_{n-1}^* \quad n > 0, & \Psi_n^* &= \psi_{-1} \psi_{-2} \cdots \psi_n \quad n < 0. \end{aligned} \quad (3.1.22)$$

Given  $g$ , which satisfy bilinear identity (3.1.12), one constructs KP and TL tau-functions (1.0.7) as follows:

$$\tau_{KP}(n, \mathbf{t}) = \langle n | e^{H(\mathbf{t})} g | n \rangle, \quad (3.1.23)$$

$$\tau_{TL}(n, \mathbf{t}, \mathbf{t}^*) = \langle n | e^{H(\mathbf{t})} g e^{H^*(\mathbf{t}^*)} | n \rangle. \quad (3.1.24)$$

If one fixes the variables  $n, \mathbf{t}^*$ , then  $\tau_{TL}$  may be recognized as  $\tau_{KP}$ .

The times  $\mathbf{t} = (t_1, t_2, \dots)$  and  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$  are called higher Toda lattice times [19, 23] (the first set  $\mathbf{t}$  is in the same time the set of higher KP times. The first times of this set  $t_1, t_2, t_3$  are independent variables for KP equation (1.0.5), which is the first nontrivial equation in the KP hierarchy).

**Fermions and Schur functions.** In the KP theory it is suitable to use the definition of the Schur function  $s_\lambda$  in terms of higher times

$$s_\lambda(\mathbf{t}) = \det(h_{n_i - i + j}(\mathbf{t}))_{1 \leq i, j \leq r}, \quad (3.1.25)$$

where  $h_m(\mathbf{t})$  is the elementary Schur polynomial defined by the Taylor's expansion:

$$e^{\xi(\mathbf{t}, z)} = \exp\left(\sum_{k=1}^{+\infty} t_k z^k\right) = \sum_{n=0}^{+\infty} z^n h_n(\mathbf{t}). \quad (3.1.26)$$

**Lemma 1** [4]

For  $-j_1 < \cdots < -j_k < 0 \leq i_s < \cdots < i_1$ ,  $s - k \geq 0$  the next formula is valid:

$$\langle s - k | e^{H(\mathbf{t})} \psi_{-j_1}^* \cdots \psi_{-j_k}^* \psi_{i_s} \cdots \psi_{i_1} | 0 \rangle = (-1)^{j_1 + \cdots + j_k + (k-s)(k-s+1)/2} s_\lambda(\mathbf{t}), \quad (3.1.27)$$

where the partition  $\lambda = (n_1, \dots, n_{s-k}, n_{s-k+1}, \dots, n_{s-k+j_1})$  is defined by the pair of partitions:

$$(n_1, \dots, n_{s-k}) = (i_1 - (s - k) + 1, i_2 - (s - k) + 2, \dots, i_{s-k}), \quad (3.1.28)$$

$$(n_{s-k+1}, \dots, n_{s-k+j_1}) = (i_{s-k+1}, \dots, i_s | j_1 - 1, \dots, j_k - 1). \quad (3.1.29)$$

The proof is achieved by direct calculation. Here  $(\dots | \dots)$  is another notation for a partition due to Frobenius (see [1]).

### 3.2 Partitions and free fermions

**Lemma 2** Given partition  $\lambda = (i_1, \dots, i_s | j_1 - 1, \dots, j_s - 1)$  let us introduce the notation

$$|\lambda\rangle = (-1)^{j_1 + \dots + j_s} \psi_{-j_1}^* \cdots \psi_{-j_s}^* \psi_{i_s} \cdots \psi_{i_1} |0\rangle \quad (3.2.1)$$

$$\langle\lambda| = (-1)^{j_1 + \dots + j_s} \langle 0 | \psi_{i_1}^* \cdots \psi_{i_s}^* \psi_{-j_s} \cdots \psi_{-j_1} \quad (3.2.2)$$

Then

$$\langle\lambda|\mu\rangle = \delta_{\lambda,\mu} \quad (3.2.3)$$

where  $\delta$  is Kronecker symbol, and

$$s_\lambda(\mathbf{H}^*)|0\rangle = |\lambda\rangle \quad (3.2.4)$$

$$\langle 0 | s_\lambda(\mathbf{H}) = \langle\lambda| \quad (3.2.5)$$

here  $s_\lambda$  is defined according to (2.2.6) where instead of  $\mathbf{t} = (t_1, t_2, \dots)$  we have

$$\mathbf{H}^* = (H_{-1}, \frac{H_{-2}}{2}, \dots, \frac{H_{-m}}{m}, \dots) \quad (3.2.6)$$

$$\mathbf{H} = (H_1, \frac{H_2}{2}, \dots, \frac{H_m}{m}, \dots) \quad (3.2.7)$$

The proof of (3.2.4) and of (3.2.5) follows from using Lemma 1 and the development

$$e^{\sum_{m=1}^{\infty} H_m t_m} = \sum_{\lambda} s_\lambda(\mathbf{H}) s_\lambda(\mathbf{t}) \quad (3.2.8)$$

**Lemma 3**

$$s_\lambda(-\mathbf{A})|0\rangle = r_\lambda(0)|\lambda\rangle \quad (3.2.9)$$

where  $s_\lambda$  is defined according to (2.2.6) where instead of  $\mathbf{t} = (t_1, t_2, \dots)$  we have

$$\mathbf{A} = (A_1, \frac{A_2}{2}, \dots, \frac{A_m}{m}, \dots) \quad (3.2.10)$$

$$A_k = \sum_{n=-\infty}^{\infty} \psi_{n-k}^* \psi_n r(n) r(n-1) \cdots r(n-k+1), \quad k = 1, 2, \dots \quad (3.2.11)$$

The proof follows from the following. First the components of vector  $\mathbf{A}$  mutually commute:  $[A_m, A_k] = 0$ . Then we apply the development

$$e^{\sum_{m=1}^{\infty} A_m t_m} = \sum_{\lambda} s_\lambda(-\mathbf{A}) s_\lambda(\mathbf{t}) \quad (3.2.12)$$

and apply the both sides to the right vacuum vector:

$$e^{\sum_{m=1}^{\infty} A_m t_m} |0\rangle = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(-\mathbf{A}) |0\rangle \quad (3.2.13)$$

Then we develop  $e^{\sum_{m=1}^{\infty} A_m t_m} |0\rangle$  using Taylor expansion of the exponential function and also use the explicit form of (3.2.11), Lemma 1, the definition of  $|\lambda\rangle$  and also (3.2.3).

### 3.3 Vacuum expectation value as a scalar product. Symmetric function theory consideration.

Let us remember the definition of scalar product (2.1.13) on the space of symmetric functions. It is known that one may present it directly in terms of the variables  $\gamma$  and the derivatives with respect to these variables:

$$\tilde{\partial} = (\partial_{\gamma_1}, \frac{1}{2}\partial_{\gamma_2}, \dots, \frac{1}{n}\partial_{\gamma_n}, \dots) \quad (3.3.1)$$

Then it is known, that due to (2.1.13) the scalar product of polynomial functions, say,  $f$  and  $g$ , may be defined as

$$\langle f, g \rangle = (f(\tilde{\partial}) \cdot g(\gamma))|_{\gamma=0} \quad (3.3.2)$$

For instance one checks that (3.3.2) results in true answer:

$$\langle \gamma_n, \gamma_m \rangle = \frac{1}{n} \delta_{n,m}, \quad \langle p_n, p_m \rangle = n \delta_{n,m} \quad (3.3.3)$$

The following statement is important

**Proposition 1** *Let scalar product is defined as in (3.3.2). Then*

$$\langle f, g \rangle = \langle 0 | f(\mathbf{H}) g(\mathbf{H}^*) | 0 \rangle \quad (3.3.4)$$

where

$$\mathbf{H} = \left( H_1, \frac{H_2}{2}, \dots, \frac{H_n}{n}, \dots \right), \quad \mathbf{H}^* = \left( H_{-1}, \frac{H_{-2}}{2}, \dots, \frac{H_{-n}}{n}, \dots \right) \quad (3.3.5)$$

It follows from the fact that higher times and derivatives with respect to higher times gives a realization of Heisenberg algebra  $H_k H_m - H_m H_k = k \delta_{k+m,0}$ :

$$\partial_{\gamma_k} \cdot m \gamma_m - m \gamma_m \cdot \partial_{\gamma_k} = k \delta_{k+m,0} \quad (3.3.6)$$

and from the comparison of

$$\partial_{\gamma_m} \cdot 1 = 0, \quad \text{Free term of } \partial_{\gamma_m} = 0 \quad (3.3.7)$$

with

$$H_m |n\rangle = 0, m > 0, \quad \langle n | H_m = 0, m < 0 \quad (3.3.8)$$

Now let us consider deformed scalar product (2.2.2)

$$\langle s_\mu, s_\lambda \rangle_{r,n} = r_\lambda(n) \delta_{\mu,\lambda} \quad (3.3.9)$$

Each polynomial function is a linear combination of Schur functions. We have the following realization of the deformed scalar product

**Proposition 2**

$$\langle f, g \rangle_{r,n} = \langle n | f(\mathbf{H}) g(-\mathbf{A}) | n \rangle \quad (3.3.10)$$

where

$$\mathbf{H} = \left( H_1, \frac{H_2}{2}, \dots, \frac{H_m}{m}, \dots \right), \quad \mathbf{A} = \left( A_1, \frac{A_2}{2}, \dots, \frac{A_m}{m}, \dots \right) \quad (3.3.11)$$

$H_k$  and  $A_k$  are defined respectively by (3.1.13) and (3.2.11).

### 3.4 KP tau-function $\tau_r(n, \mathbf{t}, \mathbf{t}^*)$

For the collection of independent variables  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$  we denote

$$A(\mathbf{t}^*) = \sum_{m=1}^{\infty} t_m^* A_m. \quad (3.4.1)$$

where  $A_m$  were defined above in (3.2.11).

Using the explicit form of  $A_m$  (3.2.11) and anti-commutation relations (3.1.1) we obtain

$$e^{A(\mathbf{t}^*)} \psi(z) e^{-A(\mathbf{t}^*)} = e^{-\xi_r(\mathbf{t}^*, z^{-1})} \cdot \psi(z) \quad (3.4.2)$$

$$e^{A(\mathbf{t}^*)} \psi^*(z) e^{-A(\mathbf{t}^*)} = e^{\xi_{r'}(\mathbf{t}^*, z^{-1})} \cdot \psi^*(z) \quad (3.4.3)$$

where operators

$$\xi_r(\mathbf{t}^*, z^{-1}) = \sum_{m=1}^{+\infty} t_m \left( \frac{1}{z} r(D) \right)^m, \quad D = z \frac{d}{dz}, \quad r'(D) = r(-D) \quad (3.4.4)$$

act on all functions of  $z$  on the right hand side according to the rule which was described in the beginning of the subsection 3.4, namely  $r(D) \cdot z^n = r(n) z^n$ . The exponents in (3.4.2), (3.4.3), (3.4.4) are considered as their Taylor series.

Therefore we have

**Lemma** The fermionic operator  $e^{A(\mathbf{t}^*)}$  solves bilinear identity (3.1.12):

$$\left[ \text{res}_{z=0} \psi(z) \otimes \psi^*(z), e^{A(\mathbf{t}^*)} \otimes e^{A(\mathbf{t}^*)} \right] = 0 \quad (3.4.5)$$

Of cause this Lemma is also a result of a general treating in [4].

Now it is natural to consider the following tau function:

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) := \langle n | e^{H(\mathbf{t})} e^{-A(\mathbf{t}^*)} | n \rangle. \quad (3.4.6)$$

Let us use the fermionic realization of scalar product (3.3.10). We immediately obtain

#### Proposition 3

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \langle e^{\sum_{m=1}^{\infty} m t_m \gamma_m}, e^{\sum_{m=1}^{\infty} m t_m^* \gamma_m} \rangle_{r,n} \quad (3.4.7)$$

Due to (2.2.5) we get

#### Proposition 4

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*). \quad (3.4.8)$$

We shall not consider the problem of convergence of this series. The variables  $M, \mathbf{t}$  play the role of KP higher times,  $\mathbf{t}^*$  is a collection of group times for a commuting subalgebra of additional symmetries of KP (see [31, 30, 10] and Remark 7 in [32]). From different point of view (3.4.8) is a tau-function of two-dimensional Toda lattice [23] with two sets of continuous variables  $\mathbf{t}, \mathbf{t}^*$  and one discrete variable  $M$ . Formula (3.4.8) is symmetric with respect to  $\mathbf{t} \leftrightarrow \mathbf{t}^*$ . For given  $r$  we define the function  $r'$ :

$$r'(n) := r(-n). \quad (3.4.9)$$

Tau function  $\tau_r$  has properties

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \tau_r(n, \mathbf{t}^*, \mathbf{t}), \quad \tau_{r'}(-n, -\mathbf{t}, -\mathbf{t}^*) = \tau_r(n, \mathbf{t}, \mathbf{t}^*). \quad (3.4.10)$$

Also  $\tau_r(n, \mathbf{t}, \mathbf{t}^*)$  does not change if  $t_m \rightarrow a^m t_m, t_m^* \rightarrow a^{-m} t_m^*, m = 1, 2, \dots$ , and it does change if  $t_m \rightarrow a^m t_m, m = 1, 2, \dots, r(n) \rightarrow a^{-1} r(n)$ . (3.4.10) follows from the relations

$$r'_{\lambda}(n) = r_{\lambda'}(-n), \quad s_{\lambda}(\mathbf{t}) = (-)^{|\lambda|} s_{\lambda'}(-\mathbf{t}). \quad (3.4.11)$$

Let us notice that tau-function (3.4.8) can be viewed as a result of action of additional symmetries [32],[16],[31] on the vacuum tau-function.

### 3.5 Algebra of $\Psi DO$ on a circle and a development of $e^{A(\mathbf{t}^*)}, e^{\tilde{A}(\mathbf{t})}$

Given functions  $r, \tilde{r}$ , the operators

$$A_m = - \sum_{n=-\infty}^{\infty} r(n) \cdots r(n-m+1) \psi_n \psi_{n-m}^*, \quad m = 1, 2, \dots, \quad (3.5.1)$$

$$\tilde{A}_m = \sum_{n=-\infty}^{\infty} \tilde{r}(n+1) \cdots \tilde{r}(n+m) \psi_n \psi_{n+m}^*, \quad m = 1, 2, \dots \quad (3.5.2)$$

belong to the Lie algebra of pseudo-differential operators on a circle with central extension. These operators may be also rewritten in the following form

$$A_m = \frac{1}{2\pi\sqrt{-1}} \oint : \psi^*(z) \left( \frac{1}{z} r(D) \right)^m \cdot \psi(z) : \quad (3.5.3)$$

$$\tilde{A}_m = -\frac{1}{2\pi\sqrt{-1}} \oint : \psi^*(z) (\tilde{r}(D)z)^m \cdot \psi(z) : \quad (3.5.4)$$

where  $D = z \frac{d}{dz}$ , pseudo-differential operators  $r(D), \tilde{r}(D)$  act on functions of  $z$  to the right. This action is given as follows:  $r(D) \cdot z^n = r(n)z^n$ , where  $z^n, n \in \mathbb{Z}$  is the basis of holomorphic functions in the punctured disk  $1 > |z| > 0$ . Central extensions of the Lie algebra of generators (3.5.3) (remember the subsection 3.1) are described by the formulae

$$\omega_n(A_m, A_k) = \delta_{mk} \tilde{r}(n+m-1) \cdots \tilde{r}(n) r(n) \cdots r(n-m+1), \quad \omega_n - \omega_{n'} \sim 0 \quad (3.5.5)$$

where  $n$  is an integer. These extension which differs by the choice of  $n$  are cohomological ones. The choice corresponds to the choice of normal ordering  $::$ , which may chosen in a different way as  $:a: := a - \langle n|a|n \rangle$ ,  $n$  is an integer. (We choose it by  $n = 0$  see subsection 3.1). When functions  $r, \tilde{r}$  are polynomials the operators  $\left(\frac{1}{z}r(D)\right)^n, (\tilde{r}(D)z)^n$  belong to the  $W_\infty$  algebra, while the fermionic operators (3.2.11), (3.5.2) belong to the algebra with central extension denoted by  $\widehat{W}_\infty$  [36].

We are interested in calculating the vacuum expectation value of different products of operators of type  $e^{\sum A_m t_m^*}$  and  $e^{\sum \tilde{A}_m t_m}$ .

**Lemma** Let partitions  $\lambda = (i_1, \dots, i_s | j_1 - 1, \dots, j_s - 1)$  and  $\mu = (\tilde{i}_1, \dots, \tilde{i}_r | \tilde{j}_1 - 1, \dots, \tilde{j}_r - 1)$  satisfy the relation  $\lambda \supseteq \mu$ . The following is valid:

$$\begin{aligned} \langle 0 | \psi_{i_1}^* \cdots \psi_{i_r}^* \psi_{-\tilde{j}_r} \cdots \psi_{-\tilde{j}_1} e^{A(\mathbf{t}^*)} \psi_{-j_1}^* \cdots \psi_{-j_s}^* \psi_{i_s} \cdots \psi_{i_1} | 0 \rangle = \\ = (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_{\lambda/\mu}(\mathbf{t}^*) r_{\lambda/\mu}(0) \end{aligned} \quad (3.5.6)$$

$$\begin{aligned} \langle 0 | \psi_{i_1}^* \cdots \psi_{i_s}^* \psi_{-j_s} \cdots \psi_{-j_1} e^{\tilde{A}(\mathbf{t})} \psi_{-\tilde{j}_1}^* \cdots \psi_{-\tilde{j}_r}^* \psi_{\tilde{i}_r} \cdots \psi_{\tilde{i}_1} | 0 \rangle = \\ = (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_{\lambda/\mu}(\mathbf{t}) \tilde{r}_{\lambda/\mu}(0) \end{aligned} \quad (3.5.7)$$

where  $s_{\lambda/\mu}(\mathbf{t})$  is a skew Schur function [1]:

$$s_{\lambda/\mu}(\mathbf{t}) = \det \left( h_{\lambda_i - \mu_j - i + j}(\mathbf{t}) \right), \quad (3.5.8)$$

and

$$r_{\lambda/\mu}(n) := \prod_{i,j \in \lambda/\mu} r(n+j-i), \quad \tilde{r}_{\lambda/\mu}(n) := \prod_{i,j \in \lambda/\mu} \tilde{r}(n+j-i) \quad (3.5.9)$$

The proof is achieved by a development of  $e^A = 1 + A + \dots$ ,  $e^{\tilde{A}} = 1 + \tilde{A} + \dots$  and the direct evaluation of vacuum expectations (3.5.6), (3.5.7), and a determinant formula for the skew Schur function of Example 22 in Sec 5 of [1].

Let us introduce vectors

$$|\lambda, n\rangle = (-1)^{j_1+\dots+j_s} \psi_{-j_1+n}^* \cdots \psi_{-j_s+n}^* \psi_{i_s+n} \cdots \psi_{i_1+n} |n\rangle \quad (3.5.10)$$

$$\langle \lambda, n| = (-1)^{j_1+\dots+j_s} \langle n| \psi_{i_1+n}^* \cdots \psi_{i_s+n}^* \psi_{-j_s+n} \cdots \psi_{-j_1+n} \quad (3.5.11)$$

As we see

$$\langle \lambda, n|\mu, m\rangle = \delta_{mn} \delta_{\lambda\mu} \quad (3.5.12)$$

Using Lemma 1 and the development (3.2.12) we get

**Lemma**

$$|\lambda, n\rangle = s_\lambda(\mathbf{H}^*)|n\rangle, \quad \langle \lambda, n| = \langle n| s_\lambda(\mathbf{H}) \quad (3.5.13)$$

and

$$s_\lambda(-\mathbf{A})|n\rangle = r_\lambda(n) s_\lambda(\mathbf{H}^*)|n\rangle, \quad \langle n| s_\lambda(\tilde{\mathbf{A}}) = \tilde{r}_\lambda(n) \langle n| s_\lambda(\mathbf{H}) \quad (3.5.14)$$

$$\mathbf{H} = \left( H_1, \frac{H_2}{2}, \dots, \frac{H_m}{m}, \dots \right), \quad \tilde{\mathbf{A}} = \left( \tilde{A}_1, \frac{\tilde{A}_2}{2}, \dots, \frac{\tilde{A}_m}{m}, \dots \right) \quad (3.5.15)$$

where  $A_m, \tilde{A}_m$  are given by (3.2.11), (3.5.2).

There are the following developments:

**Proposition 5**

$$e^{-A(\mathbf{t}^*)} = \sum_{n \in \mathbb{Z}} \sum_{\lambda \supseteq \mu} |\mu, n\rangle s_{\lambda/\mu}(\mathbf{t}^*) r_{\lambda/\mu}(n) \langle \lambda, n| \quad (3.5.16)$$

$$e^{\tilde{A}(\mathbf{t})} = \sum_{n \in \mathbb{Z}} \sum_{\lambda \supseteq \mu} |\lambda, n\rangle s_{\lambda/\mu}(\mathbf{t}) \tilde{r}_{\lambda/\mu}(n) \langle \mu, n| \quad (3.5.17)$$

where  $s_{\lambda/\mu}(\mathbf{t})$  is the skew Schur function (3.5.8) and  $r_{\lambda/\mu}(n), \tilde{r}_{\lambda/\mu}(n)$  are defined in (3.5.9).

The proof follows from the relation [1]

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle \quad (3.5.18)$$

## 4 Integral representation of the scalar product and random matrices

In case the function  $r$  has zero the scalar product (8.1.1) is degenerate one. For simplicity let us take  $r(0) = 0, r(k) \neq 0, k \leq n$ . The scalar product (8.1.1) non generate on the subspace of symmetric functions spanned by Schur functions  $\{s_\lambda, l(\lambda) \leq n\}$ ,  $l(\lambda)$  is the length of partition  $\lambda$ .

We found the integral representation of this scalar product.

To get this representation one should suggest that there exist a function  $\mu_r(xy)$  with the following properties

$$\left( y - \frac{1}{x} r(-D_x) \right) \mu_r(xy) = 0, \quad D_x = x \frac{d}{dx} \quad (4.0.19)$$

$$\left( x - \frac{1}{y} r(-D_y) \right) \mu_r(xy) = 0, \quad D_y = y \frac{d}{dy} \quad (4.0.20)$$

and the integral

$$\int_{\Gamma} \mu_r(xy) dx dy = A \quad (4.0.21)$$

is finite and does not vanish.

The integration domain  $\Gamma$  should be chosen in such a way that the operator  $\frac{1}{x}r(D)$ ,  $D = x\frac{d}{dx}$  is conjugated to the operator  $\frac{1}{x}r(-D)$ ,  $D = x\frac{d}{dx}$ , and also

$$\frac{1}{A} \int_{\Gamma} \mu_r(xy) x^n dx dy = \frac{1}{A} \int_{\Gamma} \mu_r(xy) y^n dx dy = \delta_{n,0} \quad (4.0.22)$$

At last with the help of (4.0.19) we get

$$\frac{1}{A} \int_{\Gamma} \mu_r(xy) x^n y^m dx dy = \delta_{n,m} r(1)r(2) \cdots r(n) \quad (4.0.23)$$

Now one can evaluate the integral

$$I_{\lambda,\nu} = A^{-n} \int \cdots \int \Delta(\mathbf{x}) \Delta(\mathbf{y}) s_{\lambda}(\mathbf{x}) s_{\nu}(\mathbf{y}) \prod_{k=1}^n \mu_r(x_k y_k) dx_k dy_k \quad (4.0.24)$$

where  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$

$$\Delta(\mathbf{z}) = \prod_{i < j}^n (z_i - z_j) \quad (4.0.25)$$

Using the definition of the Schur functions (2.1.10) and the definition of  $r_{\lambda}(n)$  (2.2.1) we finely get the relation

$$I_{\lambda\nu} = r_{\lambda}(n) \delta_{\nu,\lambda} \quad (4.0.26)$$

and we obtain

### Proposition 6

$$\langle f, g \rangle_{r,n} = A^{-n} \int_{\Gamma} \cdots \int_{\Gamma} \Delta(\mathbf{x}) \Delta(\mathbf{y}) f(\mathbf{x}) g(\mathbf{y}) \prod_{k=1}^n \mu_r(x_k y_k) dx_k dy_k \quad (4.0.27)$$

**In case  $\mathbf{r}(\mathbf{0}) = \mathbf{0}$**  the series

$$\mu_r(xy) = 1 + \frac{xy}{r(-1)} + \frac{(xy)^2}{r(-1)r(-2)} + \cdots \quad (4.0.28)$$

is the formal solution of (4.0.19), (4.0.20). This series is equal to the tau-function of hypergeometric type related to  $1/r'$ , where

$$r'(n) := r(-n) \quad (4.0.29)$$

$$\mu_r(xy) = \tau_{1/r'}(1, \mathbf{t}(x), \mathbf{t}^*(y)), \quad kt_k = x^k, \quad kt_k^* = y^k \quad (4.0.30)$$

Let us suggest that the series (4.0.28) is convergent (this fact depends on the choice of the function  $r$ ), and the integral (4.0.21) exists, then one may choose this  $\mu_r$  for (4.0.27).

Let us remember that if

$$\mathbf{t}(\mathbf{x}) = (t_1(\mathbf{x}), t_2(\mathbf{x}), \dots), \quad \mathbf{t}^*(\mathbf{y}) = (t_1^*(\mathbf{y}), t_2^*(\mathbf{y}), \dots), \quad mt_m = \sum_{i=1}^n x_i^m, \quad mt_m^* = \sum_{i=1}^n y_i^m \quad (4.0.31)$$



we have the following determinant representation [6] , [7] (see an Appendix below (5.6.14), (5.2.11), (5.2.12))

$$\Delta(\mathbf{x})\Delta(\mathbf{y})\tau_r(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) = c_n \det(\tau_r(1, x_i, y_k))_{i,k=1}^n, \quad c_n = \prod_{k=0}^{n-1} (r(k))^{k-n} \quad (4.0.32)$$

where  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$  which is true on the level of formal series.

$\tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y}))$  is the tau-function (3.4.8), where the function  $r$  is chosen as  $1/r'$

$$\tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) = 1 + \frac{(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{r(-1)} + \dots \quad (4.0.33)$$

Due to determinant representation (4.0.32) we see that this series is a convergent one, since we consider the series (4.0.30) to be convergent.

**Proposition 7** *Let the tau function  $\tau_{1/r'}$  of (3.4.8), where  $r'(n) \equiv r(-n)$ , is a convergent series, and let  $r(0) = 0$ . Then we have the following realization of scalar product (2.2.2)*

$$\langle f, g \rangle_{r,n} = A^{-n} \int_{\Gamma} \dots \int_{\Gamma} f(\mathbf{x})g(\mathbf{y})\tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) (\Delta(\mathbf{x}))^2 (\Delta(\mathbf{y}))^2 \prod_{k=1}^n dx_k dy_k \quad (4.0.34)$$

The proof follows from the fact that the function  $\tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y}))$  is a symmetric function of variables  $x_1, \dots, x_n$ . Then by changing notations of variables  $x_i$  inside of the integral we obtain the Proposition.

Then it follows from (4.0.34),(2.2.5) that for the same conditions the following duality relation is true

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \quad (4.0.35)$$

$$A^{-n}c_n^{-1} \int_{\Gamma} \dots \int_{\Gamma} e^{\sum_{i=1}^n \sum_{m=1}^{\infty} (t_m x_i^m + t_m^* y_i^m)} \tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) (\Delta(\mathbf{x}))^2 (\Delta(\mathbf{y}))^2 \prod_{k=1}^n dx_k dy_k \quad (4.0.36)$$

Also due to (2.2.7),(2.2.8) we have the following

$$\tau_{r_3}(0, \mathbf{t}, \mathbf{t}^*) = \quad (4.0.37)$$

$$A^{-n}c_n^{-1} \int_{\Gamma} \dots \int_{\Gamma} \tau_{r_1}(k, \mathbf{t}, \mathbf{t}^*(\mathbf{x})) \tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) \tau_{r_2}(m, \mathbf{t}(\mathbf{y}), \mathbf{t}^*) \Delta^2(\mathbf{x}) \Delta^2(\mathbf{y}) \prod_{k=1}^n dx_k dy_k \quad (4.0.38)$$

where

$$r_3(i) = r_1(k+i)r_2(m+i)r(n+i) \quad (4.0.39)$$

## 4.1 Standard scalar product

Let us suggest a realization of the standard scalar product (3.3.2). This scalar product is a particular case of (3.3.9). In this case  $r \equiv 1$ . This function has no zeroes, therefore the consideration of the previous subsection is not valid. We realize (3.3.2) as follows

$$\langle f, g \rangle = \int f(\mathbf{t}) e^{-\sum_{m=1}^{\infty} m |t_m|^2} g(\bar{\mathbf{t}}) \prod_{m=1}^{\infty} \frac{m d^2 t_m}{\pi} \quad (4.1.1)$$

where  $\bar{t}_m$  is the complex conjugated of  $t_m$ . The formula for the scalar product of two tau-functions is similar to (4.0.37):

$$\langle \tau_{r_1}(k), \tau_{r_2}(m) \rangle = \tau_{r_3}(0) = \quad (4.1.2)$$

where due to (3.4.8),(3.3.2)  $r_3(i) = r_1(k+i)r_2(m+i)$ , and

$$= \int \tau_{r_1}(k, \mathbf{t}, \gamma) e^{-\sum_{k=1}^{\infty} k |\gamma_k|^2} \tau_{r_2}(m, \bar{\gamma}, \mathbf{t}^*) \prod_{i=1}^{\infty} \frac{k d^2 \gamma_k}{\pi} \quad (4.1.3)$$

## 4.2 Normal matrix model

Let us consider a model of normal matrices. A matrix is called normal if it commutes with Hermitian conjugated matrix.

Let  $M$  be  $n$  by  $n$  normal matrix. The following integral is called the model of normal matrices:

$$I^{NM}(n, \mathbf{t}, \mathbf{t}^*; \mathbf{u}) = \int dM dM^+ e^{U(MM^+) + V_1(M) + V_2(M^+)} = \quad (4.2.1)$$

$$C \int \cdots \int |\Delta(\mathbf{z})|^2 e^{\sum_{i=1}^n U(z_i \bar{z}_i) + V_1(z_i) + V_2(\bar{z}_i)} \prod_{i=1}^n dz_i d\bar{z}_i \quad (4.2.2)$$

where  $dM = \prod_{i < k} d\Re M_{ik} d\Im M_{ik} \prod_{i=1}^N dM_{ii}$ ,  $C$  is a number which appears due to the angle integration, and  $U$  is defined by  $\mathbf{u} = (u_1, u_2, \dots)$ :

$$U(MM^+) = \sum_{n=1}^{\infty} u_n \text{Tr}(MM^+)^n, \quad V_1(M) = \sum_{m=1}^{\infty} t_m \text{Tr} M^m, \quad V_2(M) = \sum_{m=1}^{\infty} t_m^* \text{Tr} (M^+)^m \quad (4.2.3)$$

$$\Delta(\mathbf{z}) = \prod_{i < k}^n (z_i - z_k) \quad (4.2.4)$$

and  $\Delta(z) = 1$  for  $n = 1$ .

As we see the integral (4.2.2) has a form of (4.0.27), where  $\mathbf{x} = \mathbf{z}, \mathbf{y} = \bar{\mathbf{z}}$  and

$$f(\mathbf{z}) = e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m z_i^m}, \quad g(\bar{\mathbf{z}}) = e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m^* \bar{z}_i^m}, \quad \mu_r(z_i \bar{z}_i) = e^{U(z_i \bar{z}_i)} \quad (4.2.5)$$

Comparing  $e^U$  with  $\mu_r$  of (4.0.28) we obtain that the set of variables  $\mathbf{u} = (u_1, u_2, \dots)$  is related to values of  $r(n), n = -1, -2, \dots$  as follows

$$h_m(\mathbf{u}) = \frac{1}{r(-1)} \cdots \frac{1}{r(-m)}, \quad r(-m) = \frac{h_{m-1}(\mathbf{u})}{h_m(\mathbf{u})}, \quad e^{\sum_{k=1}^{\infty} u_k z^k} = \sum_{m=0}^{\infty} h_m(\mathbf{u}) z^m \quad (4.2.6)$$

Finely formula (2.2.5), where  $p_m = \sum_{i=1}^n z_i^m$ , yields the asymptotic series for (4.2.1):

$$I^{NM}(n, \mathbf{t}, \mathbf{t}^*; \mathbf{u}) = \langle e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m z_i^m}, e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m^* \bar{z}_i^m} \rangle_{r,n} = \sum_{\lambda} r_{\lambda}(n) s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) \quad (4.2.7)$$

## 4.3 Two-matrix model

Details of this and of the next section one can find in [5]. Let us evaluate the following integral over  $n$  by  $n$  matrices  $M_1$  and  $M_2$ , where  $M_1$  is a Hermitian matrix and  $M_2$  is an anti-Hermitian one

$$I^{2MM}(n, \mathbf{t}, \mathbf{t}^*) = \int e^{V_1(M_1) + V_2(M_2)} e^{-\text{Tr} M_1 M_2} dM_1 dM_2 \quad (4.3.1)$$

where

$$V_1(M_1) = \sum_{m=1}^{\infty} t_m \text{Tr} M_1^m, \quad V_2(M_2) = \sum_{m=1}^{\infty} t_m^* \text{Tr} M_2^m \quad (4.3.2)$$

It is well-known [12],[2] that this integral reduces to the integral over eigenvalues  $x_i$  and  $y_i$  of matrices  $M_1$  and  $M_2$  respectively.

To do it first we diagonalize the matrices:  $M_1 = U_1 X U_1^{-1}, M_2 = \sqrt{-1} U_2 Y U_2^{-1}$ , where  $U_1, U_2$  are unitary matrices and  $X, Y$  are diagonal matrices with real entries. Then one calculates the Jacobians of the change of variables  $M_1 \rightarrow (X, U_1), M_2 \rightarrow (Y, U_2) : dM_1 =$

$(\Delta(\mathbf{x}))^2 \prod_{k=1}^n dx_k \prod_{i < j} d\Re(U_1)_{ij} d\Im(U_1)_{ij}$  and  $dM_2 = (\Delta(\mathbf{y}))^2 \prod_{k=1}^n dy_k \prod_{i < j} d\Re(U_2)_{ij} d\Im(U_2)_{ij}$ . As a result we get

$$I^{HTMM}(n, \mathbf{t}, \mathbf{t}^*) = \int e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m x_i^m} e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m y_i^m} I(\mathbf{x}, \mathbf{y}) \Delta^2(\mathbf{x}) \Delta^2(\mathbf{y}) \prod_{k=1}^n dx_k dy_k \int d_* U_1 \quad (4.3.3)$$

where

$$I(\mathbf{x}, \mathbf{y}) = \int e^{-\text{Tr} XUYU^{-1}} dUU^{-1} \quad (4.3.4)$$

is the so-called **Harish-Chandra-Itzykson-Zuber integral**. This integral was calculated [12], [2] :

$$I(\mathbf{x}, \mathbf{y}) = \frac{\det \left( e^{-x_i y_k \sqrt{-1}} \right)_{i,k=1}^n}{\Delta(\mathbf{x}) \Delta(\mathbf{y})} \quad (4.3.5)$$

Due to (4.0.32) formula (4.3.5) is a manifest of the fact that the Harish-Chandra-Itzykson-Zuber integral actually is a tau-function (see a different approach to this fact in [41]). This is the tau-function  $\tau_r$  which corresponds to  $r(n) = -1/n$ , in other words we can write

$$\frac{I(\mathbf{x}, \mathbf{y})}{0! \cdots (n-1)!} = \tau_{1/r'}(n, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})), \quad 1/r'(n) = -\frac{1}{n} \quad (4.3.6)$$

For  $n = 1$ ,  $x_1 = x$ ,  $y_1 = y$  we have

$$\tau_{1/r'}(1, \mathbf{t}(\mathbf{x}), \mathbf{t}^*(\mathbf{y})) = e^{-xy\sqrt{-1}} \quad (4.3.7)$$

Due to the equality

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-xy\sqrt{-1}} dx dy = 2\pi \quad (4.3.8)$$

the condition (4.0.21) is fulfilled.

Therefore due to (4.0.34) we obtain that the model of two Hermitian random matrices is the series of hypergeometric type (2.2.5), where  $p_m = \sum_{i=1}^n z_i^m$  and  $r(n) = n$ :

$$I^{2MM}(n, \mathbf{t}, \mathbf{t}^*) = \frac{C_n}{(2\pi)^n} < e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m z_i^m}, e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m^* z_i^{*m}} >_{r,n} = \frac{C_n}{(2\pi)^n} \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) \quad (4.3.9)$$

where we remember that according to (2.2.1)

$$(n)_{\lambda} = \prod_{i,j \in \lambda} (n + j - i) = \frac{\Gamma(n+1+\lambda_1) \Gamma(n+\lambda_2) \cdots \Gamma(\lambda_n)}{\Gamma(n+1) \Gamma(n) \cdots \Gamma(1)} \quad (4.3.10)$$

and  $C_n$  incorporates the volume of unitary group.

The same result one can obtain with the help of (4.0.27), using the known result that integral (4.3.3) is equal to

$$C_n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m x_i^m} e^{\sum_{i=1}^n \sum_{m=1}^{\infty} t_m y_i^m} e^{-\sum_{i=1}^n x_i y_i \sqrt{-1}} \Delta(\mathbf{x}) \Delta(\mathbf{y}) \prod_{i=1}^n dx_i dy_i \quad (4.3.11)$$

where  $x_i$  and  $y_i$  are eigenvalues of  $M_1$  and  $M_2$  respectively.

This integral has a form of (4.2.5) if one takes the integrals over variables  $x_i$  be along the real axe, while the integrals over variables  $y_i$  be along the imaginary axe. Then we take  $r(n) = n$  so that

$$\mu_r(xy) = \tau_{1/r'} = 1 - xy + \frac{x^2 y^2}{2} + \cdots = e^{-xy} \quad (4.3.12)$$

$$\int_{\Re} \int_{\Im} e^{-xy} dx dy = 2\pi \quad (4.3.13)$$

Thus we have the following asymptotic perturbation series [5]

$$\frac{I^{2MM}(n, \mathbf{t}, \mathbf{t}^*)}{I^{2MM}(n, 0, 0)} = \sum_{\lambda} (n)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) \quad (4.3.14)$$

In case all higher coupling constants are equal to 0 (namely, in the case of Gauss matrix integral) this series can be easily evaluated as

$$I^{2MM}(1, t_1, t_2, 0, 0, \dots; t_1^*, t_2^*, 0, 0, \dots) = \sum_{m=0}^{\infty} m! h_m(\mathbf{t}) h_m(\mathbf{t}^*) = \frac{\exp \frac{t_1 t_1^* + t_2 (t_1^*)^2 + t_2^* (t_1)^2}{1 - 4t_2 t_2^*}}{\sqrt{1 - 4t_2 t_2^*}} \quad (4.3.15)$$

and for  $n > 1$  one can use the formula (for example see [5])

$$\tau_r(n, \mathbf{t}, \mathbf{t}^*) = \frac{1}{(n-1)! \dots 0!} \det \left( \frac{\partial^{k+m}}{\partial_{t_1}^k \partial_{t_1^*}^m} \tau_r(1, \mathbf{t}, \mathbf{t}^*) \right)_{k,m=0}^{n-1} \quad (4.3.16)$$

which is true for the case  $r(0) = 0$ .

## 4.4 Hermitian one matrix model

Let us evaluate simplest perturbation terms for one matrix model using the series (4.3.15) [5]. Let us take all  $t_k = 0$  except  $t_4$ , and all  $t_k^* = 0$  except  $t_2^*$  [42]. If we put

$$g_4 = -4N^{-1}t_4, \quad g = (2Nt_2^*)^{-1} \quad (4.4.1)$$

we obtain that

$$\sum_{\lambda} (N)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) = \int dM e^{-N \text{Tr}(\frac{g}{2} M^2 + g_4 M^4)} \quad (4.4.2)$$

We chose the normalization of matrix integral in such a way that it is equal to 1 when  $g_4 = 0$ .

**Remark.** As it is well-known, see for instance [43], according to Feynman rules for one matrix model we have (a) to each propagator (double line) is associated a factor  $1/(Ng)$  (which is  $2t_2^*$  in our notations) (b) to each four leg vertex is associated a factor  $(-Ng_4)$  (which is  $4t_4$  in our notations) (c) to each closed single line is associated a factor  $N$ . Therefore one may say that the factors  $(N)_{\lambda}$  in (4.4.2) is responsible for closed lines, the factors  $s_{\lambda}(\mathbf{t} = 0, t_2, 0, \dots)$  are responsible for propagators and the factors  $s_{\lambda}(\mathbf{t}^*)$  are responsible for vertices. As we see we get the following structure for the Feynman diagrams, containing  $k = |\lambda|/4$  vertices and  $2k = |\lambda|/2$  propagators:

$$(Ng_4)^{|\lambda|/4} (Ng)^{-|\lambda|/2} \sum_{|\lambda|=4k} (N)_{\lambda} a_{\lambda} b_{\lambda} \quad (4.4.3)$$

where the number  $|\lambda|$  is the weight of partition  $\lambda$ , and  $a_{\lambda}, b_{\lambda}$  are numbers:

$$a_{\lambda} = s_{\lambda}(0, 0, 0, 1, 0, \dots), \quad b_{\lambda} = s_{\lambda}(0, 1, 0, \dots) \quad (4.4.4)$$

Since only  $t_4$  is non vanishing the first non vanishing Schur functions correspond to  $\lambda$  of weight 4, which are  $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$ . Let us notice that two last partition are conjugated to the two first, and the partition  $(2, 2)$  is self conjugated.

To evaluate l.h.s. of (4.4.2) we use (2.2.1) , (2.1.12) and take into account the property for conjugated partition  $\lambda'$  we have

$$s_{\lambda'}(-\mathbf{t}) = (-)^{|\lambda|} s_{\lambda}(\mathbf{t}), \quad r'_{\lambda'}(-n) = r_{\lambda}(n) \quad (4.4.5)$$

where  $|\lambda|$  is the number of boxes in the Young diagram (or the same - the weight of the partition), and for  $r'$  see (3.4.9).

The partition  $\lambda = (4)$  gives

$$N(N+1)(N+2)(N+3)t_4 \frac{(t_2^*)^2}{2!} = (N^4 + 6N^3 + 11N^2 + 6N) \left( -\frac{Ng_4}{4} \right) \left( \frac{1}{8N^2g^2} \right) \quad (4.4.6)$$

Due to (4.4.5) , (4.4.1) the conjugated partition  $(1, 1, 1, 1)$  gives the same answer but  $N \rightarrow -N$ . (Thus one needs to keep only even powers of  $N$ ).

The partition  $(3, 1)$  gives

$$N(N+1)(N+2)(N-1)(-t_4) \frac{(-t_2^*)^2}{2!} = (N^4 + 2N^3 - N^2 - 2N) \left( -\frac{Ng_4}{4} \right) \left( \frac{1}{8N^2g^2} \right) \quad (4.4.7)$$

Again due to the contribution of the conjugated partition  $(2, 1, 1)$  one needs to keep only even powers of  $N$ .

First Schur function in the l.h.s of (4.4.2) is equal to zero (due to (2.1.12) and since only  $t_4$  is non vanishing).

Finely we find first perturbation terms as

$$1 - \left( \frac{N^2}{2} + \frac{1}{4} \right) \frac{g_4}{g^2} + \dots \quad (4.4.8)$$

This answer coincides with the well-known answer one gets by Feynman diagram method, see [43]

For the next order one takes partitions of weight 8, as a consequence that it is only  $t_4$  that is non vanishing. One can get that non vanishing contribution is only due to self conjugated partitions  $(3, 3, 2)$ ,  $(4, 2, 1, 1)$ , and to partitions  $(8)$ ;  $(7, 1)$ ,  $(4, 4)$ ;  $(6, 1, 1)$ ,  $(4, 3, 1)$ ;  $(5, 1, 1, 1)$  and conjugated to these ones. The enumerated partitions yields respectively  $h_4^2, h_4^2$ , then  $h_8; -h_8, h_4^2; -h_8, -h_4^2; -h_8$  for  $s_{\lambda}(\mathbf{t}^*)$ . And it yields respectively  $h_4^2 - h_4h_2^2, h_4^2 - h_4h_2^2$ , then  $h_8; -h_8, h_4^2; -h_8 - h_6h_2, -h_4^2 + h_6h_2; -h_8 + h_6h_2$  for  $s_{\lambda}(\mathbf{t})$ . For partitions  $(3, 3, 2)$ ,  $(4, 2, 1, 1)$ , and for partitions  $(8)$ ;  $(7, 1)$ ,  $(4, 4)$ ;  $(6, 1, 1)$ ,  $(4, 3, 1)$ ;  $(5, 1, 1, 1)$  we get respectively

$$N(N+1)(N+2)(N-1)(N)(N+1)(N-2)(N-1) \left( t_4^2 \right) \left( \left( \frac{(t_2^*)^2}{2!} \right)^2 - \frac{(t_2^*)^2}{2!} (t_2^*)^2 \right) \quad (4.4.9)$$

$$N(N+1)(N+2)(N+3)(N-1)(N)(N-2)(N-3) \left( t_4^2 \right) \left( \left( \frac{(t_2^*)^2}{2!} \right)^2 - \frac{(t_2^*)^2}{2!} (t_2^*)^2 \right) \quad (4.4.10)$$

and

$$N(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7) \left( \frac{t_4^2}{2!} \right) \left( \frac{(t_2^*)^4}{4!} \right) \quad (4.4.11)$$

$$N(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N-1) \left( -\frac{t_4^2}{2!} \right) \left( -\frac{(t_2^*)^4}{4!} \right) \quad (4.4.12)$$

$$N(N+1)(N+2)(N+3)(N-1)(N)(N+1)(N+2) \left(t_4^2\right) \left(\frac{(t_2^*)^2}{2!}\right)^2 \quad (4.4.13)$$

$$N(N+1)(N+2)(N+3)(N+4)(N+5)(N-1)(N-2) \left(\frac{t_4^2}{2!}\right) \left(\frac{(t_2^*)^4}{4!} - t_2^* \frac{(t_2^*)^3}{3!}\right) \quad (4.4.14)$$

$$N(N+1)(N+2)(N+3)(N-1)(N)(N+1)(N-2) \left(-t_4^2\right) \left(-\left(\frac{(t_2^*)^2}{2!}\right)^2 + t_2^* \frac{(t_2^*)^3}{3!}\right) \quad (4.4.15)$$

$$N(N+1)(N+2)(N+3)(N+4)(N-1)(N-2)(N-3) \left(-\frac{t_4^2}{2!}\right) \left(-\frac{(t_2^*)^4}{4!} + t_2^* \frac{(t_2^*)^3}{3!}\right) \quad (4.4.16)$$

At the highest order at  $N$  which is  $N^6$  one gets zero.

$$(32N^6 + 320N^4 + 488N^2)t_4^2(t_2^*)^4 = (32N^6 + 320N^4 + 488N^2) \left(\frac{Ng_4}{4}\right)^2 \left(\frac{2}{Ng}\right)^4 = \quad (4.4.17)$$

$$\frac{g_4^2}{g^4}(32N^4 + 320N^2 + 488)$$

## 5 Appendices A

### 5.1 Baker-Akhiezer functions and bilinear identities [4]

Vertex operators  $V_\infty(\mathbf{t}, z)$ ,  $V_\infty^*(\mathbf{t}, z)$  and  $V_0(\mathbf{t}^*, z)$ ,  $V_0^*(\mathbf{t}^*, z)$  act on the space  $C[t_1, t_2, \dots]$  of polynomials in infinitely many variables, and are defined by the formulae:

$$V_\infty(\mathbf{t}, z) = z^M e^{\xi(\mathbf{t}, z)} e^{-\xi(\tilde{\partial}, z^{-1})}, \quad V_\infty^*(\mathbf{t}, z) = z^{-M} e^{-\xi(\mathbf{t}, z)} e^{\xi(\tilde{\partial}, z^{-1})}, \quad (5.1.1)$$

$$V_0(\mathbf{t}^*, z) = z^M e^{-\xi(\mathbf{t}^*, z^{-1})} e^{\xi(\tilde{\partial}^*, z)}, \quad V_0^*(\mathbf{t}^*, z) = z^{-M} e^{\xi(\mathbf{t}^*, z^{-1})} e^{-\xi(\tilde{\partial}^*, z)}, \quad (5.1.2)$$

where  $\tilde{\partial} = (\frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \dots)$ ,  $\tilde{\partial}^* = (\frac{\partial}{\partial t_1^*}, \frac{1}{2} \frac{\partial}{\partial t_2^*}, \frac{1}{3} \frac{\partial}{\partial t_3^*}, \dots)$ .

We have the rules of the bosonization:

$$\langle M+1 | e^{H(\mathbf{t})} \psi(z) = V_\infty(\mathbf{t}, z) \langle M | e^{H(\mathbf{t})}, \quad \langle M-1 | e^{H(\mathbf{t})} \psi^*(z) = V_\infty^*(\mathbf{t}, z) \langle M | e^{H(\mathbf{t})} \frac{dz}{z}, \quad (5.1.3)$$

$$\psi(z) e^{H^*(\mathbf{t}^*)} |M\rangle = V_0(\mathbf{t}^*, z) e^{H^*(\mathbf{t}^*)} |M-1\rangle, \quad \psi^*(z) e^{H^*(\mathbf{t}^*)} |M\rangle = V_0^*(\mathbf{t}^*, z) e^{H^*(\mathbf{t}^*)} |M+1\rangle \frac{dz}{z} \quad (5.1.4)$$

The Baker-Akhiezer functions and conjugated Baker-Akhiezer functions are:

$$w_\infty(M, \mathbf{t}, \mathbf{t}^*, z) = \frac{V_\infty(\mathbf{t}, z) \tau}{\tau}, \quad w_\infty^*(M, \mathbf{t}, \mathbf{t}^*, z) = \frac{V_\infty^*(\mathbf{t}, z) \tau}{\tau}, \quad (5.1.5)$$

$$w_0(M, \mathbf{t}, \mathbf{t}^*, z) = \frac{V_0(\mathbf{t}^*, z) \tau(M+1)}{\tau(M)}, \quad w_0^*(M, \mathbf{t}, \mathbf{t}^*, z) = \frac{V_0^*(\mathbf{t}^*, z) \tau(M-1)}{\tau(M)}, \quad (5.1.6)$$

where

$$\tau(M) = \tau(M, \mathbf{t}, \mathbf{t}^*) = \langle M | e^{H(\mathbf{t})} g e^{H^*(\mathbf{t}^*)} | M \rangle. \quad (5.1.7)$$

Both KP and TL hierarchies are described by the bilinear identity:

$$\oint w_\infty(M, \mathbf{t}, \mathbf{t}^*, z) w_\infty^*(M', \mathbf{t}', \mathbf{t}'^*, z) dz = \oint w_0(M, \mathbf{t}, \mathbf{t}^*, z^{-1}) w_0^*(M', \mathbf{t}', \mathbf{t}'^*, z^{-1}) z^{-2} dz, \quad (5.1.8)$$

which holds for any  $\mathbf{t}, \mathbf{t}^*, \mathbf{t}', \mathbf{t}'^*$  for any integers  $M, M'$ .

The Schur functions  $s_\lambda(\mathbf{t})$  are well-known examples of tau-functions which correspond to rational solutions of the KP hierarchy. It is known that not any linear combination of Schur functions turns to be a KP tau-function, in order to find these combinations one should solve bilinear difference equation, see [19], which is actually a version of discrete Hirota equation. Below we shall present KP tau-functions which are infinite series of Schur polynomials, and which turn to be known hypergeometric functions (7.0.19),(7.0.25).

## 5.2 $H_0(\mathbf{T})$ , fermions $\psi(\mathbf{T}, z)$ , $\psi^*(\mathbf{T}, z)$ , bosonization rules [9]

Throughout this section we assume that  $r, \tilde{r} \neq 0$ .

Put

$$r(n) = e^{T_{n-1}-T_n}, \quad \tilde{r}(n) = e^{\tilde{T}_{n-1}-\tilde{T}_n} \quad (5.2.1)$$

New variables  $T_n, n \in \mathbb{Z}$  (and also  $\tilde{T}_n$ ) are defined up to a constant independent of  $n$ . We define a generator  $H_0(\mathbf{T}) \in \hat{gl}(\infty)$  :

$$H_0(\mathbf{T}) := \sum_{n=-\infty}^{\infty} T_n : \psi_n^* \psi_n :, \quad (5.2.2)$$

which produces the following transformation:

$$e^{\mp H_0(\mathbf{T})} \psi_n e^{\pm H_0(\mathbf{T})} = e^{\pm T_n} \psi_n, \quad e^{\mp H_0(\mathbf{T})} \psi_n^* e^{\pm H_0(\mathbf{T})} = e^{\mp T_n} \psi_n^* \quad (5.2.3)$$

This is the transformation which sends  $A_m, \tilde{A}_m$  of (3.2.11), (3.5.2) to  $H_{-m}, H_m$  respectively. Therefore

$$e^{H_0(\tilde{\mathbf{T}})} \tilde{A}(\mathbf{t}) e^{-H_0(\tilde{\mathbf{T}})} = H(\mathbf{t}), \quad e^{-H_0(\mathbf{T})} A(\mathbf{t}^*) e^{H_0(\mathbf{T})} = -H^*(\mathbf{t}^*). \quad (5.2.4)$$

It is convenient to consider the following fermionic operators:

$$\psi(\mathbf{T}, z) = e^{H_0(\mathbf{T})} \psi(z) e^{-H_0(\mathbf{T})} = \sum_{n=-\infty}^{n=+\infty} e^{-T_n} z^n \psi_n, \quad (5.2.5)$$

$$\psi^*(\mathbf{T}, z) = e^{H_0(\mathbf{T})} \sum_{n=-\infty}^{n=+\infty} z^{-n-1} \psi_n^* e^{-H_0(\mathbf{T})} = \sum_{n=-\infty}^{n=+\infty} e^{T_n} z^{-n-1} \psi_n^*. \quad (5.2.6)$$

Now we consider Hirota-Miwa change of variables:  $\mathbf{t} = \pm \mathbf{t}(\mathbf{x}^N)$  and  $\mathbf{t}^* = \pm \mathbf{t}^*(\mathbf{y}^N)$ .

Considering (5.1.3),(5.1.4) and using (5.2.4),(5.2.5),(5.2.6) (5.1.3),(5.1.4) we get the following bosonization rules

$$e^{-A(\mathbf{t}^*(\mathbf{y}^N))} |M\rangle = \frac{\psi(\mathbf{T}, y_1) \cdots \psi(\mathbf{T}, y_N) |M - N\rangle}{\Delta^+(M, N, \mathbf{T}, \mathbf{y}^N)}, \quad (5.2.7)$$

$$e^{-A(-\mathbf{t}^*(\mathbf{y}^N))} |M\rangle = \frac{\psi^*(\mathbf{T}, y_1) \cdots \psi^*(\mathbf{T}, y_N) |M + N\rangle}{\Delta^-(M, N, \mathbf{T}, \mathbf{y}^N)}, \quad (5.2.8)$$

$$\langle M | e^{\tilde{A}(\mathbf{t}(\mathbf{x}^N))} = \frac{\langle M - N | \psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_N}) \cdots \psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_1})}{\tilde{\Delta}^+(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N)}, \quad (5.2.9)$$

$$\langle M | e^{\tilde{A}(-\mathbf{t}(\mathbf{x}^N))} = \frac{\langle M + N | \psi(-\tilde{\mathbf{T}}, \frac{1}{x_N}) \cdots \psi(-\tilde{\mathbf{T}}, \frac{1}{x_1})}{\tilde{\Delta}^-(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N)}. \quad (5.2.10)$$

The coefficients are

$$\Delta^\pm(M, N, \mathbf{T}, \mathbf{y}^N) = \frac{\prod_{i < j}^N (y_i - y_j)}{(y_1 \cdots y_N)^{N \mp M}} \times \frac{\tau(M, \mathbf{0}, \mathbf{T}, \mathbf{0})}{\tau(M \mp N, \mathbf{0}, \mathbf{T}, \mathbf{0})}, \quad (5.2.11)$$

$$\tilde{\Delta}^\pm(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N) = \frac{\prod_{i < j}^N (x_i - x_j)}{(x_1 \cdots x_N)^{N-1 \mp M}} \times \frac{\tau(M, \mathbf{0}, \tilde{\mathbf{T}}, \mathbf{0})}{\tau(M \mp N, \mathbf{0}, \tilde{\mathbf{T}}, \mathbf{0})}, \quad (5.2.12)$$

where in case  $N = 1$  the Vandermond products  $\prod_{i < j}^N (y_i - y_j)$ ,  $\prod_{i < j}^N (x_i - x_j)$  are replaced by 1. The origin of the notation  $\tau(M, \mathbf{0}, \mathbf{T}, \mathbf{0})$  is explained latter, see (5.3.5), (5.3.7), and used for

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{T_{n-1} + \cdots + T_1 + T_0} = e^{nT_0} \prod_{k=0}^{n-1} (r(k))^{k-n}, \quad n > 0, \quad (5.2.13)$$

$$\tau(0, \mathbf{0}, \mathbf{T}, \mathbf{0}) = 1, \quad n = 0, \quad (5.2.14)$$

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{-T_n - \cdots - T_{-2} - T_{-1}} = e^{nT_0} \prod_{k=0}^{-n-1} (r(k))^{k+n}, \quad n < 0. \quad (5.2.15)$$

Therefore in Hirota-Miwa variables one can rewrite the vacuum expectation of the exponents of (3.5.1) and (3.5.2):

$$\frac{\langle M | e^{\tilde{A}(\mathbf{t}(\mathbf{x}^N))} e^{-A(\mathbf{t}^*(\mathbf{y}^N))} | M \rangle}{\tilde{\Delta}^+(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N) \Delta^+(M, N, \mathbf{T}, \mathbf{y}^N)} = \frac{\langle M - N | \psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_N}) \cdots \psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_1}) \psi(\mathbf{T}, y_1) \cdots \psi(\mathbf{T}, y_N) | M - N \rangle}{\tilde{\Delta}^+(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N) \Delta^+(M, N, \mathbf{T}, \mathbf{y}^N)}, \quad (5.2.16)$$

$$\frac{\langle M | e^{\tilde{A}(-\mathbf{t}(\mathbf{x}^N))} e^{-A(-\mathbf{t}^*(\mathbf{y}^N))} | M \rangle}{\tilde{\Delta}^-(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N) \Delta^-(M, N, \mathbf{T}, \mathbf{y}^N)} = \frac{\langle M + N | \psi(-\tilde{\mathbf{T}}, \frac{1}{x_N}) \cdots \psi(-\tilde{\mathbf{T}}, \frac{1}{x_1}) \psi^*(\mathbf{T}, y_1) \cdots \psi^*(\mathbf{T}, y_N) | M + N \rangle}{\tilde{\Delta}^-(M, N, \tilde{\mathbf{T}}, \mathbf{x}^N) \Delta^-(M, N, \mathbf{T}, \mathbf{y}^N)}. \quad (5.2.17)$$

This representation is suitable for an application of Wick rule (3.1.6), see (5.6.14) below.

### 5.3 Toda lattice tau-function $\tau(n, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)$ [9]

Here we consider *Toda lattice* (TL) [15], [4], [23]. Our notations  $M, \mathbf{t}, \mathbf{t}^*$  correspond to the notations  $s, x, -y$  respectively in [23].

Let us consider a special type TL tau-function (3.1.24), which depends on the three sets of variables  $\mathbf{t}, \mathbf{T}, \mathbf{t}^*$  and on  $M \in \mathbb{Z}$ :

$$\tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = \langle M | e^{H(\mathbf{t})} \exp \left( \sum_{n=-\infty}^{\infty} T_n : \psi_n^* \psi_n : \right) e^{H^*(\mathbf{t}^*)} | M \rangle, \quad (5.3.1)$$

where  $: \psi_n^* \psi_n := \psi_n^* \psi_n - \langle 0 | \psi_n^* \psi_n | 0 \rangle$ . Since the operator  $\sum_{n=-\infty}^{\infty} : \psi_n^* \psi_n :$  commutes with all elements of the  $\widehat{gl}(\infty)$  algebra, one can put  $T_{-1} = 0$  in (5.3.1). With respect to the KP and the TL dynamics the variables  $T_n$  have a meaning of integrals of motion.

As we shall see the hypergeometric functions (7.0.10), (7.0.12), (7.0.19), (7.0.16) listed in the Appendix are ratios of tau-functions (5.3.1) evaluated at special values of times  $M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*$ . It is true only in the case when all parameters  $a_k$  of the hypergeometric functions are non integers. For the case when at least one of the indices  $a_k$  is an integer, we will need a tau-function of an open Toda chain which will be considered in the Appendix.

Tau-function (5.3.1) is linear in each  $e^{T_n}$ . It is described by the Proposition



**Proposition 8**

$$\frac{\tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{\tau(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} = 1 + \sum_{\lambda \neq \mathbf{0}} e^{(T_{M-1}-T_{n_1+M-1})+(T_{M-2}-T_{n_2+M-2})+\dots+(T_{M-l}-T_{n_l+M-l})} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{t}^*). \quad (5.3.2)$$

The sum is going over all different partitions

$$\lambda = (n_1, n_2, \dots, n_l), \quad l = 1, 2, 3, \dots, \quad (5.3.3)$$

excluding the partition  $\mathbf{0}$ .

Let  $r \neq 0, \tilde{r} \neq 0$ . Then we put

$$r(n) = e^{T_{n-1}-T_n}, \quad \tilde{r}(n) = e^{\tilde{T}_{n-1}-\tilde{T}_n}. \quad (5.3.4)$$

to show the equivalence of (3.4.8) and (5.3.2) in this case. We have

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{-T_{n-1}-\dots-T_1-T_0} = e^{-nT_0} \prod_{k=0}^{n-1} (r(k))^{n-k}, \quad n > 0, \quad (5.3.5)$$

$$\tau(0, \mathbf{0}, \mathbf{T}, \mathbf{0}) = 1, \quad n = 0, \quad (5.3.6)$$

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{T_n+\dots+T_{-2}+T_{-1}} = e^{-nT_0} \prod_{k=0}^{-n-1} (r(k))^{-k-n}, \quad n < 0. \quad (5.3.7)$$

**Proposition 9** Let  $\tau(n, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)$  is the Toda lattice tau function (5.3.1). Then

$$\frac{\tau(n, \mathbf{t}, \tilde{\mathbf{T}} + \mathbf{T}, \mathbf{t}^*)}{\tau(n, \mathbf{0}, \tilde{\mathbf{T}} + \mathbf{T}, \mathbf{0})} = \langle n | e^{\tilde{A}(\mathbf{t})} e^{-A(\mathbf{t}^*)} | n \rangle = \tau_{\tilde{r}r}(n, \mathbf{t}, \mathbf{t}^*). \quad (5.3.8)$$

where

$$A(\mathbf{t}^*) = \sum_{k=1}^{\infty} A_k t_k^*, \quad \tilde{A}(\mathbf{t}) = \sum_{k=1}^{\infty} \tilde{A}_k t_k, \quad (5.3.9)$$

$A_k, \tilde{A}_k$  are defined by (3.5.3), (3.5.4). The functions  $\tilde{r}, r$  are related to  $\tilde{\mathbf{T}}, \mathbf{T}$  by (5.3.4), the product  $r\tilde{r}$  has no zeroes at integer values of it's argument.

This proposition follows from formulas (5.2.3)-(5.2.4).

We shall consider  $\tilde{r} = 1$ .

The notation  $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$  will be used only for the KP tau-function (3.4.6). Notation  $\tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)$  is used for Toda lattice tau-function. These tau-functions are related via (??). KP tau-function  $\tau(n) := \tau_r(n, \mathbf{t}, \mathbf{t}^*)$  obeys the following Hirota equation:

$$\tau(n) \partial_{t_1^*} \partial_{t_1} \tau(n) - \partial_{t_1} \tau(n) \partial_{t_1^*} \tau(n) = r(n) \tau(n-1) \tau(n+1). \quad (5.3.10)$$

The equation

$$\partial_{t_1} \partial_{t_1^*} \phi_n = r(n) e^{\phi_{n-1}-\phi_n} - r(n+1) e^{\phi_n-\phi_{n+1}} \quad (5.3.11)$$

which is similar to the Toda lattice equation holds for

$$\phi_n(\mathbf{t}, \mathbf{t}^*) = -\log \frac{\tau_r(n+1, \mathbf{t}, \mathbf{t}^*)}{\tau_r(n, \mathbf{t}, \mathbf{t}^*)} \quad (5.3.12)$$

Equations (5.3.11) and (5.3.10) are still true in case  $r$  has zeroes. If the function  $r$  has no integer zeroes, using the change of variables

$$\varphi_n = -\phi_n - T_n, \quad (5.3.13)$$

we obtain Toda lattice equation in the standard form [23]:

$$\partial_{t_1} \partial_{t_1^*} \varphi_n = e^{\varphi_{n+1} - \varphi_n} - e^{\varphi_n - \varphi_{n-1}}. \quad (5.3.14)$$

As we see from (5.3.13) the variables  $T_n$  might have the meaning of asymptotic values of the fields  $\phi_n$  for the class of tau-functions (5.3.1) which is characterized by the property  $\varphi_n \rightarrow 0$  as  $t_1 \rightarrow 0$ .

## 5.4 Expressions and linear equations for tau-functions I (Hirota-Miwa variables)

It is well-known fact that tau-functions solves bilinear equations. The class of tau-functions under our consideration solves also linear equations.

The relations of this subsection correspond to the case when at least one set of variables (either  $\mathbf{t}$  or  $\mathbf{t}^*$  or both ones) is substituted with Hirota-Miwa variables.

In cases  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  (using (5.2.9) and (3.4.3)) and  $\mathbf{t} = -\mathbf{t}(\mathbf{x}^N)$  (using (5.2.10) and (3.4.2)) one gets the following representations

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \Delta^{-1} \left( e^{\xi_{r'}(\mathbf{t}^*, x_1)} \dots e^{\xi_{r'}(\mathbf{t}^*, x_N)} \cdot \Delta \right), \quad \Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \dots x_N)^{N-1-M}} \quad (5.4.1)$$

$$\tau_r(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \Delta^{-1} \left( e^{-\xi_r(\mathbf{t}^*, x_1)} \dots e^{-\xi_r(\mathbf{t}^*, x_N)} \cdot \Delta \right), \quad \Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \dots x_N)^{N-1+M}} \quad (5.4.2)$$

where the exponents are formal series  $e^\xi = 1 + \xi + \dots$ , and  $r(D_i) \prod_k x_k^{a_k} = r(a_i) \prod_k x_k^{a_k}$ . Let us write down that according to (3.4.4), (3.4.9)

$$\xi_{r'}(\mathbf{t}^*, x_i) = \sum_{m=1}^{+\infty} t_m^* (x_i r(D_i))^m, \quad \xi_r(\mathbf{t}^*, x) = \sum_{m=1}^{+\infty} t_m^* (x_i r(-D_i))^m, \quad D_i = x_i \frac{\partial}{\partial x_i} \quad (5.4.3)$$

Let us note that  $[\xi_{r'}(\mathbf{t}^*, x_n), \xi_{r'}(\mathbf{t}^*, x_m)] = 0 = [\xi_r(\mathbf{t}^*, x_n), \xi_r(\mathbf{t}^*, x_m)]$  for all  $n, m$ .

(5.4.2) may be also obtained from (5.4.1) with the help of (3.4.10).

From (5.4.1) it follows for  $m = 1, 2, \dots$  that

$$\left( \frac{\partial}{\partial t_m^*} - \sum_{i=1}^N (x_i r(D_{x_i}))^m \right) (\Delta \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*)) = 0, \quad \Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \dots x_N)^{N-1-M}}. \quad (5.4.4)$$

In case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N), \mathbf{t}^* = \mathbf{t}^*(\mathbf{y}^{N^*})$  we get

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^{N^*})) = \sum_{\lambda \in P} r_{\mathbf{n}}(M) s_{\lambda}(\mathbf{x}^N) s_{\lambda}(\mathbf{y}^{N^*}) \quad (5.4.5)$$

$$= \frac{1}{\Delta(x)} \prod_{i=1}^N \prod_{j=1}^{N^*} (1 - y_j x_i r(D_{x_i}))^{-1} \cdot \Delta(x) \quad (5.4.6)$$

$$= \frac{1}{\Delta(y)} \prod_{i=1}^N \prod_{j=1}^{N^*} (1 - x_i y_j r(D_{y_j}))^{-1} \cdot \Delta(y) \quad (5.4.7)$$

where

$$\Delta(x) = \frac{\prod_{i < j}^N (x_i - x_j)}{(x_1 \cdots x_N)^{N-1-M}}, \quad \Delta(y) = \frac{\prod_{i < j}^{N^*} (y_i - y_j)}{(y_1 \cdots y_N)^{N^*-1-M}} \quad (5.4.8)$$

Looking at (5.4.6), (5.4.7) one derives the following system of linear equations

$$\left( D_{y_j} - \sum_{i=1}^N \frac{1}{1 - y_j x_i r(D_{x_i})} + N \right) (\Delta(x) \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^{N^*}))) = 0, \quad j = 1, \dots, N^* \quad (5.4.9)$$

$$\left( D_{x_i} - \sum_{j=1}^{N^*} \frac{1}{1 - x_i y_j r(D_{y_j})} + N^* \right) (\Delta(y) \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^{N^*}))) = 0, \quad i = 1, \dots, N \quad (5.4.10)$$

In some examples below we shall take specify  $\mathbf{t}^*$  as follows.

(1) Let us take

$$mt_m = \sum_{k=1}^N x_k^m, \quad \mathbf{t}^* = (1, 0, 0, \dots) \quad (5.4.11)$$

Then one consider the equation (5.4.19) with  $k = 1$  (taking  $t_1^* = 1$ ):

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \frac{1}{\Delta} e^{x_1 r(D_1) + \dots + x_N r(D_N)} \cdot \Delta, \quad (5.4.12)$$

where

$$\Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-M-1}} \quad (5.4.13)$$

and  $D_i = x_i \partial_{x_i}$ . We get

$$\left( \sum_{i=1}^N D_i - \frac{1}{\Delta} \left( \sum_{i=1}^N x_i r(D_i) \right) \Delta \right) \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = 0 \quad (5.4.14)$$

(2) Let us take

$$mt_m = \sum_{k=1}^N x_k^m, \quad mt_m^* = \frac{z^m}{1 - q^m}, \quad m = 1, 2, \dots \quad (5.4.15)$$

Let us note that the tau-function  $\tau_r(\mathbf{t}, \mathbf{t}^*)$  will not change if instead of (5.4.27) one take

$$mt_m = \sum_{k=1}^N (zx_k)^m, \quad mt_m^* = \frac{1}{1 - q^m}, \quad m = 1, 2, \dots \quad (5.4.16)$$

This gives the representation as follows

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \frac{1}{\Delta(x)} \prod_{i=1}^N \frac{1}{(x_i r(D_{x_i}); q)_\infty} \cdot \Delta(x) \quad (5.4.17)$$

We see that

$$\left( q^{\frac{N^2-N}{2}-NM} q^{D_1+\dots+D_N} - \prod_{i=1}^N (1 - x_i r(D_i)) \right) (\Delta(x) \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*)) = 0 \quad (5.4.18)$$

We shall write down more linear equations, which follow from the explicit fermionic representation of the tau-function (3.4.6) via the bosonization formulae (5.2.16) and (5.2.17). These equations may be also viewed as the constraint which result in the string equations (5.13.4) and (5.13.5).

For the variables  $\mathbf{x}^N$  in case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$ , one can use the relation  $\langle M|A_m = 0$ , and makes profit of the relation  $A_m = e^{H_0} H_{-m} e^{-H_0}$  inside the fermionic vacuum expectation value (5.2.16). This way we get the partial differential equations for the tau-function (3.4.8):

$$\frac{\partial \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*)}{\partial t_m^*} = \frac{1}{\Delta} \left( \sum_{i=1}^N (x_i r(D_{x_i}))^m \right) \Delta \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*), \quad m = 1, 2, 3, \dots, \quad (5.4.19)$$

where  $\Delta$  is proportional to  $\tilde{\Delta}^+$  of (5.2.12):

$$\Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-1-M}}. \quad (5.4.20)$$

In variables  $\mathbf{y}^{(\infty)}$ ,  $\mathbf{t}^* = -\mathbf{t}^*(\mathbf{y}^{(\infty)})$ , we can rewrite (5.4.19):

$$\begin{aligned} (-1)^k \sum_{i=1}^{+\infty} \frac{e_{k-1} \left( \frac{1}{y_1}, \dots, \frac{1}{y_{i-1}}, \frac{1}{y_{i+1}}, \dots \right)}{\prod_{j \neq i} (1 - \frac{y_i}{y_j})} \frac{\partial \tau_r(M, -\mathbf{t}(\mathbf{x}^N), -\mathbf{t}^*(\mathbf{y}^{(\infty)}))}{\partial y_i} = \\ \frac{1}{\tilde{\Delta}} \left( \sum_{i=1}^N (x_i r(-D_{x_i}))^k \right) \tilde{\Delta} \tau_r(M, -\mathbf{t}(\mathbf{x}^N), -\mathbf{t}^*(\mathbf{y}^{(\infty)})), \end{aligned} \quad (5.4.21)$$

where  $e_k(\mathbf{y})$  is a symmetric function defined through the relation  $\prod_{i=1}^{+\infty} (1 + ty_i) = \sum_{k=0}^{+\infty} t^k e_k(\mathbf{y})$ .

Also we have

$$\begin{aligned} \left( \sum_{k=1}^{M+N-1} k - \sum_{i=1}^N D_{x_i} \right) \tilde{\Delta}^- \tau(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{T}, -\mathbf{t}^*(\mathbf{y}^{(N')})) \Delta^- = \\ \left( \sum_{k=1}^{M+N'-1} k - \sum_{i=1}^{N'} \left( \frac{1}{y_i} D_{y_i} y_i \right) \right) \tilde{\Delta}^- \tau(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{T}, -\mathbf{t}^*(\mathbf{y}^{(N')})) \Delta^-, \end{aligned} \quad (5.4.22)$$

where  $\tilde{\Delta}^- = \tilde{\Delta}^-(M, N, \mathbf{0}, \mathbf{x}^N)$  and  $\Delta^- = \Delta^-(M, N, \mathbf{0}, \mathbf{y}^{(N')})$ . This formula is obtained by the insertion of the fermionic operator  $res_z : \psi^*(z) z \frac{d}{dz} \psi(z) :$  inside the fermionic vacuum expectation.

These formulae can be also written in terms of higher KP and TL times, with the help of vertex operator action, see the *Subsection "Vertex operator action"*. Then the relations (5.4.19) are the infinitesimal version of (5.5.6), while the relation (5.4.22) is the infinitesimal version of (5.5.7).

In some examples below we shall take specify  $\mathbf{t}^*$  as follows. (1) Let us take

$$mt_m = \sum_{k=1}^N x_k^m, \quad \mathbf{t}^* = (t_1^*, 0, 0, \dots) \quad (5.4.23)$$

One notes that that the tau-function  $\tau_r(\mathbf{t}, \mathbf{t}^*)$  will not change if instead of (5.4.27) one take

$$mt_m = \sum_{k=1}^N (t_1 x_k)^m, \quad \mathbf{t}^* = (1, 0, 0, \dots) \quad (5.4.24)$$

Then one consider the equation (5.4.19) with  $k = 1$ , taking  $t_1 = 1$ . We put  $D_i = x_i \partial_{x_i}$  and get

$$\left( \sum_{i=1}^N D_i - \frac{1}{\Delta} \left( \sum_{i=1}^N x_i r(D_i) \right) \Delta \right) \tau_r(\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = 0 \quad (5.4.25)$$

where  $r$  is given by (5.10.1), and

$$\Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-M-1}} \quad (5.4.26)$$

(2) Let us take

$$mt_m = \sum_{k=1}^N x_k^m, \quad mt_m^* = \frac{z^m}{1 - q^m}, \quad m = 1, 2, \dots \quad (5.4.27)$$

Let us note that the tau-function  $\tau_r(\mathbf{t}, \mathbf{t}^*)$  will not change if instead of (5.4.27) one take

$$mt_m = \sum_{k=1}^N (zx_k)^m, \quad mt_m^* = \frac{1}{1 - q^m}, \quad m = 1, 2, \dots \quad (5.4.28)$$

Then we apply the operator  $(1 - q_z^D)$ ,  $D_z = z \frac{d}{dz}$  to  $e^{-A(\mathbf{t}^*)}|0\rangle$  ( $\mathbf{t}^*$  of (5.4.27)) and get  $e^{-A(\mathbf{t}^*)}(1 - \sum_{n=0}^{+\infty} \psi_{-n}^* \psi_0)|0\rangle = e^{-A(\mathbf{t}^*)}(\sum_{n=1}^{+\infty} \psi_{-n}^* \psi_0)|0\rangle = e^{-A(\mathbf{t}^*)}(A_1 + \dots)|0\rangle$ , where terms denoted by dots vanish inside the vacuum expectation  $\langle 0|e^{H(\mathbf{t}(\mathbf{x}^N))}e^{-A(\mathbf{t}^*)}A_1|0\rangle$ . Then one consider the equation (5.4.19) with  $k = 1$ , taking  $z = 1$ . We put  $D_i = x_i \partial_{x_i}$  and because get

$$\left( \sum_{i=1}^N (1 - q^{D_1 + \dots + D_N}) - \frac{1}{\Delta} \left( \prod_{i=1}^N (1 - x_i r(D_i)) \right) \Delta \right) \tau_r(\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = 0 \quad (5.4.29)$$

where  $r$  is given by (5.10.1), and

$$\Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-M-1}} \quad (5.4.30)$$

The simplest way to derive linear equation (5.4.19), (5.4.25), (5.4.29) is to use the representations (5.4.1), (5.4.2).

## 5.5 The vertex operator action. Linear equations for tau-functions II

Now we present relations between hypergeometric functions which follow from the soliton theory, for instance see [10], [30]. (In [9] there were few misprints in the formulae presented below). Let us introduce the operators which act on functions of  $\mathbf{t}$  variables:

$$\Omega_r^{(\infty)}(\mathbf{t}, \mathbf{t}^*) := -\frac{1}{2\pi\sqrt{-1}} \lim_{\epsilon \rightarrow 0} \oint V_\infty^*(\mathbf{t}, z + \epsilon) \xi_r(\mathbf{t}^*, z^{-1}) V_\infty(\mathbf{t}, z) \frac{dz}{z}, \quad (5.5.1)$$

$$\Omega_r^{(0)}(\mathbf{t}^*, \mathbf{t}) := \frac{1}{2\pi\sqrt{-1}} \lim_{\epsilon \rightarrow 0} \oint V_0^*(\mathbf{t}^*, z + \epsilon) \xi_r^{(0)}(\mathbf{t}, z) V_0(\mathbf{t}^*, z) \frac{dz}{z}, \quad (5.5.2)$$

where  $V_\infty(\mathbf{t}, z)$ ,  $V_\infty^*(\mathbf{t}, z)$ ,  $V_0(\mathbf{t}^*, z)$ ,  $V_0^*(\mathbf{t}^*, z)$  are defined by (5.1.1), and

$$\xi_r(\mathbf{t}^*, z^{-1}) = \sum_{m=1}^{+\infty} t_m^* \left( \frac{1}{z} r(D) \right)^m, \quad \xi_r^{(0)}(\mathbf{t}, z) = \sum_{m=1}^{+\infty} t_m (r(D)z)^m \quad (5.5.3)$$

For instance

$$r = 1 : \quad \Omega_r^{(\infty)}(\mathbf{t}, \mathbf{t}^*) = \Omega_r^{(0)}(\mathbf{t}^*, \mathbf{t}) = \sum_{n>0} n t_n t_n^*. \quad (5.5.4)$$

Also we consider

$$Z_{nn}(\mathbf{t}) = -\frac{1}{4\pi^2} \oint \frac{z^n}{z^{*n}} V_\infty^*(\mathbf{t}, z^*) V_\infty(\mathbf{t}, z) \frac{dz dz^*}{z z^*}, \quad Z_{nn}^*(\mathbf{t}^*) = -\frac{1}{4\pi^2} \oint \frac{z^n}{z^{*n}} V_0^*(\mathbf{t}^*, z^*) V_0(\mathbf{t}^*, z) \frac{dz dz^*}{z z^*}. \quad (5.5.5)$$

The bosonization formulae (5.1.3), (5.1.4) result in

**Proposition 10** *We have shift argument formulae for the tau function (3.4.8):*

$$e^{\Omega_r^{(\infty)}(\mathbf{t}, \gamma)} \cdot \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau_r(M, \mathbf{t}, \mathbf{t}^* + \gamma), \quad e^{\Omega_r^{(0)}(\mathbf{t}^*, \gamma)} \cdot \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau_r(M, \mathbf{t} + \gamma, \mathbf{t}^*). \quad (5.5.6)$$

Also we have

$$e^{\sum_{n=-\infty}^{\infty} \gamma_n Z_{nn}(\mathbf{t})} \cdot \tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = e^{\sum_{n=-\infty}^{\infty} \gamma_n Z_{nn}^*(\mathbf{t}^*)} \cdot \tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = \tau(M, \mathbf{t}, \mathbf{T} + \gamma, \mathbf{t}^*), \quad (5.5.7)$$

For instance

$$e^{\Omega_r^{(\infty)}(\mathbf{t}, \mathbf{t}^*)} \cdot 1 = e^{\Omega_r^{(0)}(\mathbf{t}^*, \mathbf{t})} \cdot 1 = \tau_r(M, \mathbf{t}, \mathbf{t}^*) \quad (5.5.8)$$

Also

$$e^{\sum_{n=-\infty}^{\infty} T_n Z_{nn}} \cdot \exp\left(\sum_{n=1}^{\infty} n t_n t_n^*\right) = e^{\sum_{n=-\infty}^{\infty} T_n Z_{nn}^*} \cdot \exp\left(\sum_{n=1}^{\infty} n t_n t_n^*\right) = \tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*). \quad (5.5.9)$$

## 5.6 Determinant formulae I (Hirota-Miwa variables) [9]

In the case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  one can apply Wick theorem [19] to obtain a determinant formulae.

**Proposition 11** *A generalization of Milne's determinant formula*

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \frac{\det\left(x_i^{N-k} \tau_r(M - k + 1, \mathbf{t}(x_i), \mathbf{t}^*)\right)_{i,k=1}^N}{\det\left(x_i^{N-k}\right)_{i,k=1}^N}. \quad (5.6.1)$$

**Proof**

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) = \langle M | e^{H(\mathbf{t}(\mathbf{x}^N))} e^{-A(\mathbf{t}^*)} | M \rangle = \quad (5.6.2)$$

$$\frac{x_1^{N-M-1} \cdots x_N^{N-M-1}}{\prod_{i < j} (x_i - x_j)} \langle M | \psi_{M-1} \cdots \psi_{M-N} \psi^*\left(\frac{1}{x_N}\right) \cdots \psi^*\left(\frac{1}{x_1}\right) e^{-A(\mathbf{t}^*)} | M \rangle = \quad (5.6.3)$$

$$\frac{(x_1 \cdots x_N)^{N-M-1}}{\prod_{i < j} (x_i - x_j)} \det\left(\langle M | \psi_{M-k} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} | M \rangle\right)_{i,k=1}^N = \quad (5.6.4)$$

$$\frac{\det\left(x_i^{N-k} \tau_r(M - k + 1, \mathbf{t}(x_i), \mathbf{t}^*)\right)_{i,k=1}^N}{\det\left(x_i^{N-k}\right)_{i,k=1}^N} \quad (5.6.5)$$

Last equality follows from:

$$\langle M | \psi_{M-k} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} | M \rangle = \quad (5.6.6)$$

$$= \langle M | \psi_{M-1} \cdots \psi_{M-k+1} \psi_{M-k} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} \psi_{M-k+1}^* \cdots \psi_{M-1}^* | M \rangle + \quad (5.6.7)$$

$$+ \sum_{j=1}^{k-1} a_j^k(\mathbf{t}^*) \langle M | \psi_{M-1} \cdots \psi_{M-k+j} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} \psi_{M-k+j+1}^* \cdots \psi_{M-1}^* | M \rangle = \quad (5.6.8)$$

$$= \langle M - k + 1 | \psi_{M-k} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} | M - k + 1 \rangle + \quad (5.6.9)$$

$$+ \sum_{j=1}^{k-1} a_j^k(\mathbf{t}^*) \langle M - k + 1 + j | \psi_{M-k+j} \psi^*\left(\frac{1}{x_i}\right) e^{-A(\mathbf{t}^*)} | M - k + 1 + j \rangle = \quad (5.6.10)$$

$$= x_i^{M-k+1} \tau_r(M - k + 1, \mathbf{t}(x_i), \mathbf{t}^*) + \sum_{j=1}^{k-1} a_j^k(\mathbf{t}^*) x_i^{M-k+1+j} \tau_r(M - k + 1 + j, \mathbf{t}(x_i), \mathbf{t}^*) \quad (5.6.11)$$

Where the functions  $a_j^k(\mathbf{t}^*)$  must be derived as the results of action of operator  $e^{-A(\mathbf{t}^*)}$  on the fermions  $\psi_{M-1}, \dots, \psi_{M-k}$ . Thus we have:

$$x_i^{N-M-1} \langle M | \psi_{M-k} \psi^* \left( \frac{1}{x_i} \right) e^{-A(\mathbf{t}^*)} | M \rangle = \quad (5.6.12)$$

$$= x_i^{N-k} \tau_r(M-k+1, \mathbf{t}(x_i), \mathbf{t}^*) + \sum_{l=1}^{k-1} a_{k-l}^k(\mathbf{t}^*) x_i^{N-l} \tau_r(M-l+1, \mathbf{t}(x_i), \mathbf{t}^*) \quad (5.6.13)$$

In the case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N), \mathbf{t}^* = \mathbf{t}^*(\mathbf{y}^N)$  we apply to (5.2.16) the Wick theorem to obtain a different Proposition

**Proposition 12** For  $r \neq 0$  we take a tau function  $\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^N))$  and apply Wick's theorem. We get the determinant formula:

$$\tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^N)) = \frac{c_M}{c_{M-N}} \frac{\det(\tau_r(M-N+1, x_i, y_j))_{i,j=1}^N}{\Delta(\mathbf{x}^N) \Delta(\mathbf{y}^N)}, \quad (5.6.14)$$

where

$$c_n = \prod_{k=0}^{n-1} (r(k))^{k-n} \quad (5.6.15)$$

$$\tau_r(M-N+1, x_i, y_j) = 1 + r(M-N+1)x_i y_j + r(M-N+1)r(M-N+2)x_i^2 y_j^2 + \dots \quad (5.6.16)$$

## 5.7 Determinant formulae II

There is also determinant formulae for tau functions in variables  $\mathbf{t}, \mathbf{t}^*$ . These one we get in case the function  $r$  has zeroes. If  $r(M_0) = 0$  and  $n > 0$  then

$$\tau_r(M_0 + n, \mathbf{t}, \mathbf{t}^*) = c_n \det \left( \frac{\partial^{a+b}}{\partial t_1^a \partial t_1^{*b}} \tau_r(M_0 + 1, \mathbf{t}, \mathbf{t}^*) \right)_{a,b=0}^{n-1} \quad (5.7.1)$$

$$c_n = \prod_{k=0}^{n-1} (r(k))^{k-n} \quad (5.7.2)$$

## 5.8 Integral representations I (Hirota-Miwa variables) [9]

In the case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N), \mathbf{t}^* = \mathbf{t}^*(\mathbf{y}^N)$  we get an integral representation formulae. For the fermions (5.2.5) we easily get the relations:

$$\int \psi(\mathbf{T}, \alpha z) d\mu(\alpha) = \psi(\mathbf{T} + \mathbf{T}(\mu), z), \quad \int \psi^* \left( -\mathbf{T}, \frac{1}{\alpha z} \right) d\tilde{\mu}(\alpha) = \psi^* \left( -\mathbf{T} - \mathbf{T}(\tilde{\mu}), \frac{1}{z} \right) \quad (5.8.1)$$

where  $\mu, \tilde{\mu}$  are some integration measures, and shifts of times  $T_n$  are defined in terms of the moments:

$$\int \alpha^n d\mu(\alpha) = e^{-T_n(\mu)}, \quad \int \alpha^n d\tilde{\mu}(\alpha) = e^{-T_n(\tilde{\mu})}. \quad (5.8.2)$$

Therefore thanks to the bosonization formulae (5.2.16) we have the relations for the tau-function; below  $\mathbf{t}^*$  is defined via Hirota-Miwa variables.

**Proposition 13** *Integral representation formula holds*

$$\begin{aligned} & \int \tilde{\Delta}_{\tilde{\mathbf{T}}}(\tilde{\alpha}\mathbf{x}^N) \frac{\tau(M, \mathbf{t}(\tilde{\alpha}\mathbf{x}^N), \mathbf{T} + \tilde{\mathbf{T}}, \mathbf{t}(\alpha\mathbf{y}_{(N)}))}{\tau(M, \mathbf{0}, \mathbf{T} + \tilde{\mathbf{T}}, \mathbf{0})} \Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)}) \prod_{i=1}^N d\tilde{\mu}(\tilde{\alpha}_i) \prod_{i=1}^N d\mu(\alpha_i) \\ &= \tilde{\Delta}_{\tilde{\mathbf{T}} + \tilde{\mathbf{T}}(\tilde{\mu})}(\mathbf{x}^N) \frac{\tau(M, \mathbf{t}(\mathbf{x}^N), \mathbf{T} + \tilde{\mathbf{T}} + \tilde{\mathbf{T}}(\tilde{\mu}) + \mathbf{T}(\mu), \mathbf{t}(\mathbf{y}^N))}{\tau(M, \mathbf{0}, \mathbf{T} + \tilde{\mathbf{T}} + \tilde{\mathbf{T}}(\tilde{\mu}) + \mathbf{T}(\mu), \mathbf{0})} \Delta_{\mathbf{T} + \mathbf{T}(\mu)}(\mathbf{y}^N). \end{aligned} \quad (5.8.3)$$

where  $\Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)}) = \Delta^+(M, N, \mathbf{T}, \alpha\mathbf{y}_{(N)})$ ,  $\tilde{\Delta}_{\tilde{\mathbf{T}}}(\tilde{\alpha}\mathbf{x}^N) = \tilde{\Delta}^+(M, N, \tilde{\mathbf{T}}, \tilde{\alpha}\mathbf{x}^N)$ ,  $\alpha\mathbf{y}_{(N)} = (\alpha_1 y_1, \alpha_2 y_2, \dots, \alpha_N y_N)$  and  $\tilde{\alpha}\mathbf{x}^N = (\tilde{\alpha}_1 x_1, \tilde{\alpha}_2 x_2, \dots, \tilde{\alpha}_N x_N)$ . In particular

$$\begin{aligned} & \int \frac{\tau(M, \mathbf{t}, \mathbf{T}, \mathbf{t}(\alpha\mathbf{y}_{(N)}))}{\tau(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} \Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)}) \prod_{i=1}^N d\mu(\alpha_i) = \\ & \frac{\tau(M, \mathbf{t}, \mathbf{T} + \mathbf{T}(\mu), \mathbf{t}(\mathbf{y}^N))}{\tau(M, \mathbf{0}, \mathbf{T} + \mathbf{T}(\mu), \mathbf{0})} \Delta_{\mathbf{T} + \mathbf{T}(\mu)}(\mathbf{y}^N). \end{aligned} \quad (5.8.4)$$

Remember that arbitrary linear combination of tau-functions is not a tau-function. Formulae (5.8.3) and also (5.8.4) give the integral representations for the tau-function (3.4.6). It may help to express a tau-function with the help of a more simple one.

We shall use the following integrals

$$\frac{1}{2\pi\sqrt{-1}} \int_C \psi^*(-\mathbf{T}, \frac{\alpha}{x}) e^{\alpha} \alpha^{-b} d\alpha = \psi^*(-\mathbf{T} - \mathbf{T}^b, \frac{1}{x}), \quad (5.8.5)$$

$$\int_0^\infty \psi^*(-\mathbf{T}, \frac{1}{\alpha x}) e^{-\alpha} \alpha^{a-1} d\alpha = \psi^*(-\mathbf{T} - \mathbf{T}^a, \frac{1}{x}), \quad (5.8.6)$$

$$\frac{1}{\Gamma(b-a)} \int_0^1 \psi^*(-\mathbf{T}, \frac{1}{\alpha x}) \alpha^{a-1} (1-\alpha)^{b-a-1} d\alpha = \psi^*(-\mathbf{T} - \mathbf{T}^c, \frac{1}{x}), \quad (5.8.7)$$

where  $C$  in (5.8.5) starts at  $-\infty$  on the real axis, circles the origin in the counterclockwise direction and returns to the starting point, and

$$T_n^b = \ln \Gamma(b+n+1), \quad T_n^a = -\ln \Gamma(a+n+1), \quad T_n^c = \ln \frac{\Gamma(b+n+1)}{\Gamma(a+n+1)\Gamma(b-a)}. \quad (5.8.8)$$

Let us remind that  $q$ -versions of exponential function, of gamma function and of beta function are known as follows [21]

$$\exp_q(x) = \frac{1}{(x(1-q); q)_\infty}, \quad \Gamma_q(x) = (1-q)^{1-a} \frac{(q; q)_\infty}{(q^x; q)_\infty}, \quad B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \quad (5.8.9)$$

Gamma and beta functions have the  $q$ -integral representations

$$\Gamma_q(x) = -q^{-x} \int_0^\infty \frac{\alpha^{x-1}}{\exp_q(\alpha)} d_q \alpha, \quad B_q(x, y) = \int_0^1 \alpha^{x-1} \frac{(\alpha q; q)_\infty}{(\alpha q^y; q)_\infty} d_q \alpha \quad (5.8.10)$$

where  $q$ -integral are defined as

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{j=-\infty}^{j=\infty} q^j f(q^j), \quad \int_0^c f(x) d_q x = c(1-q) \sum_{j=0}^{j=\infty} q^j f(cq^j) \quad (5.8.11)$$



The following  $q$ -integrals will be of use:

$$\frac{q^{-a}}{q-1} \int_0^\infty \psi^* \left( -\mathbf{T}, \frac{q}{\alpha(1-q)x} \right) \frac{\alpha^{a-1}}{\exp_q(\alpha)} d_q \alpha = \psi^* \left( -\mathbf{T} - \mathbf{T}(a, q), \frac{1}{x} \right), \quad (5.8.12)$$

$$\frac{1}{\Gamma_q(b-a)} \int_0^1 \psi^* \left( -\mathbf{T}, \frac{1}{\alpha x} \right) \alpha^{a-1} \frac{(\alpha q; q)_\infty}{(\alpha q^{b-a}; q)_\infty} d_q \alpha = \psi^* \left( -\mathbf{T} - \mathbf{T}(a, b, q), \frac{1}{x} \right). \quad (5.8.13)$$

where, as one can check using (5.8.10),

$$T_n(a, q) = \ln \frac{1}{(1-q)^n \Gamma_q(a+n+1)}, \quad T_n(a, b, q) = \ln \frac{\Gamma_q(b+n+1)}{\Gamma_q(a+n+1)}. \quad (5.8.14)$$

In the same way one can consider Hirota-Miwa variables (1.0.8). In the *Examples* below we shall present hypergeometric functions listed in the *Subsections 1.2 and 1.3* as tau-functions of the type (5.2.16). Then we are able to write down integration formulae, namely (5.8.4), which express  ${}_{p+1}\Phi_s$  and  ${}_{p+1}\Phi_{s+1}$  in terms of  ${}_p\Phi_s$  with the help of (5.8.12), (5.8.13) and (5.8.3). In [34] different integral representation formula was presented, which was based on the  $q$ -analog of Selberg's integral of Askey and Kadell. By taking the limit  $q \rightarrow 1$  one can consider functions  ${}_pF_s$ . Using (5.8.8), one can express  ${}_{p+1}\mathcal{F}_s$ ,  ${}_{p+1}\mathcal{F}_{s+1}$  and  ${}_p\mathcal{F}_{s+1}$  as integrals of  ${}_p\mathcal{F}_s$  with the help of (5.8.6), (5.8.7) and (5.8.5) respectively.

## 5.9 Integral representations II

Now let us consider the case when  $\tau_r$  depends on  $\mathbf{t}, \mathbf{t}^*$ . Using (3.4.7) and (4.0.27) one gets the integral representation for  $\tau_r$  in case the function  $r$  has zero. Let us again choose  $r(0) = 0$ . Then we have the representation described above

### Proposition 14

$$\tau_r(M, \mathbf{t}, \mathbf{t}^*) = \int \cdots \int \Delta(z) \Delta(z^*) \prod_{k=1}^M e^{\sum_{n=1}^\infty (z_k^n t_n + z_k^{*n} t_n^*)} \mu_r(z_k z_k^*) dz_k dz_k^* \quad (5.9.1)$$

where

$$\Delta(z) = \prod_{i < j}^M (z_i - z_j), \quad \Delta(z^*) = \prod_{i < j}^M (z_i^* - z_j^*) \quad (5.9.2)$$

and  $\mu_r$  is defined by (4.0.28), (4.0.21).

Now it is interesting to notice that if the series  $\tau_{1/r}$  (which defines  $\mu_r$ , see (4.0.28)) is a convergent series then the series  $\tau_r$  may be divergent one. In this case one can consider the integral  $I_r(M, \mathbf{t}, \mathbf{t}^*)$  is an analog of Borel sum for the divergent series  $\tau_r$ , see [5].

## 5.10 Examples of hypergeometric series [9]

The main point of this subsection is the observation that if  $r(D)$  is a rational function of  $D$  then  $\tau_r$  is a hypergeometric series. If  $r(D)$  is a rational function of  $q^D$  we obtain  $q$ -deformed hypergeometric series. Now let us consider various  $r(D)$ .

**Example 1** Let all parameters  $b_k$  be non integers.

$${}_p r_s(D) = \frac{(D+a_1)(D+a_2) \cdots (D+a_p)}{(D+b_1)(D+b_2) \cdots (D+b_s)}. \quad (5.10.1)$$

For the vacuum expectation value (3.4.8) we have:

$${}^p\tau_r^s(M, \mathbf{t}, \mathbf{t}^*) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) \frac{(a_1 + M)_{\lambda} \cdots (a_p + M)_{\lambda}}{(b_1 + M)_{\lambda} \cdots (b_s + M)_{\lambda}}. \quad (5.10.2)$$

If in formula (5.10.2) we put

$$t_1^* = 1, \quad t_i^* = 0, \quad i > 1, \quad (5.10.3)$$

then  $s_{\lambda}(\mathbf{t}^*) = H_{\lambda}^{-1}$ , and we obtain the hypergeometric function related to Schur functions [21] (see [1] for help):

$$\begin{aligned} {}^p\tau_r^s(M, \mathbf{t}, \mathbf{t}^*) &= {}^pF_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| t_1, t_2, \dots \right) = \\ &= \sum_{\lambda} \frac{(a_1 + M)_{\lambda} \cdots (a_p + M)_{\lambda}}{(b_1 + M)_{\lambda} \cdots (b_s + M)_{\lambda}} \frac{s_{\lambda}(\mathbf{t})}{H_{\lambda}}. \end{aligned} \quad (5.10.4)$$

In the last formula  $H_{\lambda}$  is the following hook product (compare with (7.0.23)):

$$H_{\lambda} = \prod_{(i,j) \in \lambda} h_{ij}, \quad h_{ij} = (n_i + n'_j - i - j + 1). \quad (5.10.5)$$

We obtain ordinary hypergeometric function of one variable of type

$${}_{p-1}F_s(a_2 \pm 1, \dots, a_p \pm 1; b_1 \pm 1, \dots, b_s \pm 1; \pm t_1 t_1^*) = \tau_r(\pm 1, \mathbf{t}, \mathbf{T}, \mathbf{t}^*), \quad (5.10.6)$$

if we take  $a_1 = 0$ ,  $\mathbf{t} = (t_1, 0, 0, \dots)$ ,  $\mathbf{t}^* = (t_1^*, 0, 0, \dots)$ .

Now we take  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  the formula (5.10.4) turns out to be

$$\begin{aligned} {}^p\tau_r^s(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*) &= {}^pF_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N \right) = \\ &= \sum_{\substack{\lambda \\ l(\lambda) \leq N}} \frac{(a_1 + M)_{\lambda} \cdots (a_p + M)_{\lambda}}{(b_1 + M)_{\lambda} \cdots (b_s + M)_{\lambda}} \frac{s_{\lambda}(\mathbf{x}^N)}{H_{\lambda}}. \end{aligned} \quad (5.10.7)$$

We got the hypergeometric function (7.0.25) related to zonal polynomials for the symmetric space  $GL(N, C)/U(N)$  [22]. Here  $x_i = z_i^{-1}$ ,  $i = 1, \dots, N$  are the eigenvalues of the matrix  $\mathbf{X}$ , and for zonal spherical polynomials there is the following matrix integral representation

$$Z_{\lambda}(\mathbf{X}) = Z_{\lambda}(\mathbf{I}_N) \int_{U(N, C)} \Delta^{\lambda}(U^* \mathbf{X} U) d_* U, \quad (5.10.8)$$

where  $\Delta^{\lambda}(\mathbf{X}) = \Delta_1^{n_1 - n_2} \Delta_2^{n_2 - n_3} \cdots \Delta_N^{n_N}$  and  $\Delta_1, \dots, \Delta_N$  are main minors of the matrix  $\mathbf{X}$ ,  $d_* U$  is the invariant measure on  $U(N, C)$ , see [22].

Due to (5.4.1) hypergeometric function (5.10.4) has the representation as follows

$${}^pF_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N \right) = \frac{1}{\Delta} \exp(x_1 r(D_1)) \cdots \exp(x_N r(D_N)) \cdot \Delta, \quad (5.10.9)$$

where  $D_i = x_i \frac{\partial}{\partial x_i}$  and

$$\Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-M-1}} \quad (5.10.10)$$

(See also (5.12.13) for different representation)

Now let us take  $\mathbf{t}^* = (t_1^*, 0, 0, \dots)$ . Then we get the same function (5.10.7), but with each  $x_i$  changed by  $-t_1^* x_i$ . Then one consider the equation (5.4.19) with  $k = 1$ . We put  $D_i = x_i \partial_{x_i}$  and get

$$\left( \sum_{i=1}^N D_i - \frac{1}{\Delta} \left( \sum_{i=1}^N x_i r(D_i) \right) \Delta \right) {}_pF_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N \right) = 0 \quad (5.10.11)$$

where  $r$  is given by (5.10.1), and  $\Delta$  see in (5.10.10).

Taking  $N = 1$  we obtain the ordinary hypergeometric function of one variable  $x = x_1$ , which is (compare with (5.10.6)):

$${}_pF_s(a_1 + M, \dots, a_p + M; b_1 + M, \dots, b_s + M; x) = x^{-M} e^{xr(D)} \cdot x^M, \quad D = x \frac{d}{dx} \quad (5.10.12)$$

The ordinary hypergeometric series satisfies the known hypergeometric equation (we put  $M = 0$ ) written in the following form:

$$(\partial_x - {}_p r_s(D)) {}_pF_s(a_1, \dots, a_p; b_1, \dots, b_s; x) = 0, \quad D := x \partial_x. \quad (5.10.13)$$

This relation helps us to understand the meaning of function  $r$ .

It is known that the series (5.10.7) diverges if  $p > s + 1$  (until any of  $a_i + M$  is non positive integer). In case  $p = s + 1$  it converges in certain domain in the vicinity of  $\mathbf{x}^N = \mathbf{0}$ . For  $p < s + 1$  the series (5.10.7) converges for all  $\mathbf{x}^N$ . These known facts (see [21]) can be also obtained with the help of the determinant representation (5.6.1) and properties of (7.0.10).

Now we have the following representations:

$${}_{p+1}F_s \left( \begin{matrix} a, a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| x_1, \dots, x_N \right) = \frac{1}{\prod_{i < j}^N (x_i - x_j)} \int_0^\infty \dots \int_0^\infty (5.10.14)$$

$$(\alpha_1 \dots \alpha_N)^{a-N} e^{-\alpha_1 - \dots - \alpha_N} {}_pF_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| \alpha_1 x_1, \dots, \alpha_N x_N \right) \prod_{i < j}^N (\alpha_i x_i - \alpha_j x_j) d\alpha_1 \dots d\alpha_N$$

Hankel type of representation

$${}_pF_{s+1} \left( \begin{matrix} a_1, \dots, a_p \\ b, b_1, \dots, b_s \end{matrix} \middle| x_1, \dots, x_N \right) = \frac{1}{\prod_{i < j}^N (x_i - x_j)} \frac{1}{(2\pi\sqrt{-1})^N} \int_C \dots \int_C (5.10.15)$$

$$(\alpha_1 \dots \alpha_N)^{1-N-b} e^{\alpha_1 + \dots + \alpha_N} {}_pF_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| \frac{x_1}{\alpha_1}, \dots, \frac{x_N}{\alpha_N} \right) \prod_{i < j}^N (\alpha_i x_i - \alpha_j x_j) d\alpha_1 \dots d\alpha_N$$

where  $C$  starts at  $-\infty$  on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

Beta type representation:

$${}_{p+1}F_{s+1} \left( \begin{matrix} a, a_1, \dots, a_p \\ b, b_1, \dots, b_s \end{matrix} \middle| x_1, \dots, x_N \right) = \frac{1}{\prod_{i < j}^N (x_i - x_j)} \left( \frac{1}{\Gamma(b-a)} \right)^N \int_0^1 \dots \int_0^1 (5.10.16)$$

$${}_pF_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| \alpha_1 x_1, \dots, \alpha_N x_N \right) \prod_{i < j}^N (\alpha_i x_i - \alpha_j x_j) \prod_{i=1}^N \alpha_i^{a-N} (1 - \alpha_i)^{b-a-1} d\alpha_i$$

**Example 2.** Let us fix a number  $N$ . In order to get a hypergeometric function of two sets of variables  $\mathbf{x}^N, \mathbf{y}^N$  we put

$${}_p r_s(n) = \frac{\prod_{i=1}^p (a_i + n)}{\prod_{i=1}^s (b_i + n)} \frac{1}{N - M + n}, \quad (5.10.17)$$

Notice that  $r$  explicitly depends on  $N$ .

Taking  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  and  $\mathbf{t}^* = \mathbf{t}^*(\mathbf{y}^N)$ , and using (5.12.7) (see III of [1]) (see III of [1]) we obtain the formula (7.0.25)

$$\begin{aligned} \langle M | e^{H(\mathbf{t}(\mathbf{x}^N))} e^{-A(\mathbf{t}^*(\mathbf{y}^N))} | M \rangle &= {}_p \mathcal{F}_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N, \mathbf{y}^N \right) = \\ &= \sum_{\substack{\lambda \\ \iota(\lambda) \leq N}} \frac{s_\lambda(\mathbf{x}^N) s_\lambda(\mathbf{y}^N)}{(N)_\lambda} \frac{(a_1 + M)_\lambda \cdots (a_p + M)_\lambda}{(b_1 + M)_\lambda \cdots (b_s + M)_\lambda}. \end{aligned} \quad (5.10.18)$$

Due to (5.4.8) hypergeometric function (5.10.18) has the representation as follows

$${}_p \mathcal{F}_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N, \mathbf{y}^N \right) = \frac{1}{\Delta} \prod_{i,j=1}^N (1 - y_j x_i r(D_{x_i}))^{-1} \cdot \Delta \quad (5.10.19)$$

Here and below the inverse operator  $(1 - y_j x_i r(D_{x_i}))^{-1}$  is a formal series  $1 + y_j x_i r(D_{x_i}) + \cdots$ .

For example when  $N = 1$  we get the ordinary hypergeometric function of one variable  $xy = x_1 y_1$ :

$${}_p F_s(a_1 + M, \dots, a_p + M; b_1 + M, \dots, b_s + M; xy) = x^{-M} (1 - xy r(D_x))^{-1} \cdot x^M \quad (5.10.20)$$

Looking at (5.10.9) one derives the following linear equation

$$\left( D_{y_j} - \sum_{i=1}^N \frac{1}{1 - y_j x_i r(D_{x_i})} + N \right) \cdot \left( \Delta {}_p \mathcal{F}_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N, \mathbf{y}^N \right) \right) = 0 \quad (5.10.21)$$

**Example 3** The  $q$ -generalization of the *Example 3* is Milne's hypergeometric function of the single set of variables. To get it we choose

$${}_p r_s^{(q)}(n) = \frac{\prod_{i=1}^p (1 - q^{a_i+n})}{\prod_{i=1}^s (1 - q^{b_i+n})}. \quad (5.10.22)$$

Taking  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$ ,  $\mathbf{t}^* = \mathbf{t}^*(\mathbf{y}_{(\infty)})$  and putting

$$y_k = q^{k-1}, \quad k = 1, 2, \dots, \quad t_m^* = \sum_{k=1}^{+\infty} \frac{y_k^m}{m} = \frac{1}{m(1 - q^m)}, \quad m = 1, 2, \dots \quad (5.10.23)$$

we get Milne's hypergeometric function (7.0.19):

$$\begin{aligned} \langle M | e^{H(\mathbf{t})} e^{-A(\mathbf{t}^*)} | M \rangle &= {}_p \Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}^N \right) = \\ &= \sum_{\substack{\lambda \\ \iota(\lambda) \leq N}} \frac{(q^{a_1+M}; q)_\lambda \cdots (q^{a_p+M}; q)_\lambda}{(q^{b_1+M}; q)_\lambda \cdots (q^{b_s+M}; q)_\lambda} \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(\mathbf{x}^N). \end{aligned} \quad (5.10.24)$$

According to (5.4.8),(5.10.23) we have the following representation

$$\begin{aligned} {}_p\Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}^N \right) &= \frac{1}{\Delta(x)} \prod_{i=1}^N \frac{1}{(x_i r(D_{x_i}); q)_\infty} \cdot \Delta(x) \\ &= \frac{1}{\Delta} \exp_q(x_1 r(D_1)) \cdots \exp_q(x_N r(D_N)) \cdot \Delta \end{aligned} \quad (5.10.25)$$

(the notations  $(b; q)_\infty$ ,  $\exp_q(\alpha)$  see in (7.0.13), (5.8.9),  $\Delta$  is the same as in (5.10.10)). (See also (5.12.15) for different representation).

For this hypergeometric functions we have the linear equation (5.4.29) (where we choose  $r$  due to (5.10.22)), which is  $q$ -difference equation. For instance for the case  $N = 1, M = 0$  we get (compare it with (5.10.13))

$$\left( \frac{1}{x} (1 - q^D) - {}_p r_s^{(q)}(D) \right) {}_p\Phi_s(a_1, \dots, a_p; b_1, \dots, b_s; q, x) = 0, \quad D := x \partial_x, \quad (5.10.26)$$

where  ${}_p r_s^{(q)}(D)$  is defined by (5.10.22). Milne's function itself for  $N = 1$  is the ordinary basic hypergeometric function:

$${}_p\Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, x \right) = \sum_{n=0}^{+\infty} \frac{(q^{a_1+M}; q)_n \cdots (q^{a_p+M}; q)_n}{(q^{b_1+M}; q)_n \cdots (q^{b_s+M}; q)_n} \frac{x^n}{(q; q)_n}, \quad x = x_1 \quad (5.10.27)$$

Let us take  $M = 0$ . We have the following integral representation

$$\begin{aligned} {}_{p+1}\Phi_s \left( \begin{matrix} a, a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| q, x_1, \dots, x_N \right) &= \frac{(-1)^N q^{\frac{N-N^2}{2} - Na}}{(1-q)^{\frac{3N-N^2}{2}}} \frac{1}{\prod_{i < j}^N (x_i - x_j)} \int_0^\infty \cdots \int_0^\infty \\ &\quad \prod_{i < j}^N (\alpha_i x_i - \alpha_j x_j) {}_p\Phi_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| q, \frac{\alpha_1(1-q)x_1}{q}, \dots, \frac{\alpha_N(1-q)x_N}{q} \right) \prod_{i=1}^N \frac{\alpha_i^{a-1}}{\exp_q(\alpha_i)} d_q \alpha_i \end{aligned} \quad (5.10.28)$$

Also we have the Beta-type representation:

$$\begin{aligned} {}_{p+1}\Phi_{s+1} \left( \begin{matrix} a, a_1, \dots, a_p \\ b, b_1, \dots, b_s \end{matrix} \middle| q, x_1, \dots, x_N \right) &= \left( \frac{1}{\Gamma_q(b-a)} \right)^N \frac{1}{\prod_{i < j}^N (x_i - x_j)} \int_0^1 \cdots \int_0^1 \\ &\quad \prod_{i < j}^N (\alpha_i x_i - \alpha_j x_j) {}_p\Phi_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| q, \alpha_1 x_1, \dots, \alpha_N x_N \right) \prod_{i=1}^N \alpha_i^{a-1} \frac{(\alpha_i q; q)_\infty}{(\alpha_i q^{b-a}; q)_\infty} d_q \alpha_i \end{aligned} \quad (5.10.29)$$

**Example 4.** To obtain Milne's hypergeometric function of two sets of variables  $\mathbf{x}^N, \mathbf{y}^N$ , we take  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  and  $\mathbf{t}^* = \mathbf{t}^*(\mathbf{y}^N)$ . This choice restricts the sum over partitions  $\lambda$  with  $l(\lambda) \leq N$ . We put

$${}_p r_s^{(q)}(n) = \frac{\prod_{i=1}^p (1 - q^{a_i+n})}{\prod_{i=1}^s (1 - q^{b_i+n})} \frac{1}{1 - q^{N-M+n}}, \quad (5.10.30)$$

$$e^{-T_n} = \frac{1}{(1-q)^n \Gamma_q(n + N - M + 1)} \frac{\prod_{i=1}^p (1-q)^n \Gamma_q(a_i + n + 1)}{\prod_{i=1}^s (1-q)^n \Gamma_q(b_i + n + 1)}, \quad (5.10.31)$$

$$\Gamma_q(a) = (1-q)^{1-a} \frac{(q; q)_\infty}{(q^a, q)_\infty}, \quad (q^a, q)_n = (1-q)^n \frac{\Gamma_q(a+n)}{\Gamma_q(a)}. \quad (5.10.32)$$

Here  $\Gamma_q(a)$  is a  $q$ -deformed Gamma-function

$$\Gamma_q(a) = (1-q)^{1-a} \frac{(q; q)_\infty}{(q^a, q)_\infty}, \quad (q^a, q)_n = (1-q)^n \frac{\Gamma_q(a+n)}{\Gamma_q(a)}. \quad (5.10.33)$$

Using (5.12.7) (see III of [1]) we obtain the Milne's formula (7.0.26)

$$\begin{aligned} \tau_r(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*(\mathbf{y}^N)) &= {}_p\Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}^N, \mathbf{y}^N \right) = \\ &= \sum_{\substack{\lambda \\ l(\lambda) \leq N}} \frac{q^{n(\lambda)}}{H_\lambda(q)} \frac{s_\lambda(\mathbf{x}^N) s_\lambda(\mathbf{y}^N)}{s_\lambda(1, q, \dots, q^{N-1})} \frac{(q^{a_1+M}; q)_\lambda \dots (q^{a_p+M}; q)_\lambda}{(q^{b_1+M}; q)_\lambda \dots (q^{b_s+M}; q)_\lambda}. \end{aligned} \quad (5.10.34)$$

This is the KP tau-function (but not the TL one because (5.10.30) depends on TL variable  $M$ ).

We have the same type of representation (5.10.19) and the same form of linear equation (5.10.21), however we shall use (5.10.30) for  $r$  in formulae (5.10.19), (5.10.21).

To receive the basic hypergeometric function of one set of variables we must put indeterminates  $\mathbf{y}^N$  in (5.10.24) as  $y_i = q^{i-1}, i = (1, \dots, N)$ . Thus we have

$$\begin{aligned} {}_p\Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}^N \right) &= \\ &= \sum_{\substack{\lambda \\ l(\lambda) \leq N}} \frac{(q^{a_1+M}; q)_\lambda \dots (q^{a_p+M}; q)_\lambda}{(q^{b_1+M}; q)_\lambda \dots (q^{b_s+M}; q)_\lambda} \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(\mathbf{x}^N). \end{aligned} \quad (5.10.35)$$

And for  $N = 1$  we have the ordinary  $q$ -deformed hypergeometric function:

$${}_p\Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, x, y \right) = \sum_{n=0}^{+\infty} \frac{(q^{a_1+M}; q)_n \dots (q^{a_p+M}; q)_n}{(q^{b_1+M}; q)_n \dots (q^{b_s+M}; q)_n} \frac{(xy)^n}{(q; q)_n}, \quad xy = x_1 y_1 \quad (5.10.36)$$

## 5.11 Baker-Akhiezer functions and the elements of Sato Grassmannian related to the tau function of hypergeometric type

This subsection is based on [9]. Let us write down the expression for Baker-Akhiezer functions (5.1.5) in terms of Hirota-Miwa variables (1.0.8):

$$w_\infty(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*, \frac{1}{z}) = \frac{\tau_r(M, -\mathbf{t}(\mathbf{x}^{N+1}), \mathbf{t}^*)}{\tau_r(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*)} \prod_{i=1}^N (1 - \frac{x_i}{z}), \quad \mathbf{x}^{N+1} = (x_1, \dots, x_N, z), \quad (5.11.1)$$

$$w_\infty^*(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*, \frac{1}{z}) = -\frac{\tau_r(M, \mathbf{t}(\mathbf{x}^N) + [\mathbf{z}], \mathbf{t}^*)}{\tau_r(M, \mathbf{x}^N, \mathbf{t}^*)} \prod_{i=1}^N \frac{1}{(1 - \frac{x_i}{z})} \frac{dz}{z}, \quad [\mathbf{z}] = (z, \frac{z^2}{2}, \dots). \quad (5.11.2)$$

We see that the variables  $x_k, k = 1, \dots, N$  are zeroes of  $w_\infty(z)$  and poles of  $w_\infty^*(z)$ .

**Remark 1** *The associated linear problems for Baker-Akhiezer functions are read as*

$$(\partial_{t_1} - \partial_{t_1} \phi_n) w(n, \mathbf{t}, \mathbf{t}^*, z) = w(n+1, \mathbf{t}, \mathbf{t}^*, z), \quad (5.11.3)$$

$$\partial_{t_1^*} w(n, \mathbf{t}, \mathbf{t}^*, z) = r(n) e^{\phi_{n-1} - \phi_n} w(n-1, \mathbf{t}, \mathbf{t}^*, z). \quad (5.11.4)$$

where  $w$  is either  $w_\infty$  or  $w_0$ . The compatibility of these equations gives rise to the equation (5.3.11). Taking into account the second eq.(5.3.11), equations (5.11.3), (5.11.4) may be also viewed as the recurrent equations for the tau-functions which depend on different number of variables  $\mathbf{x}^N$ .

Let us write down a plane of Baker-Akhiezer functions (5.1.5), which characterizes elements of Sato Grassmannian related to the tau-function (5.3.1)  $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$ . We take  $x_k = 0, k = 1, \dots, N$  in (5.11.1) and obtain:

$$w_\infty(n, \mathbf{0}, \mathbf{t}^*, z) = z^n (1 + \sum_{m=1}^{\infty} r(n)r(n-1) \cdots r(n-m+1) h_m(-\mathbf{t}^*) z^{-m}), \quad n = M, M+1, M+2, \dots \quad (5.11.5)$$

The dual plane is

$$w_\infty^*(n, \mathbf{0}, \mathbf{t}^*, z) = z^{-n} (1 + \sum_{m=1}^{\infty} r(n)r(n+1) \cdots r(n+m-1) h_m(\mathbf{t}^*) z^{-m}) dz, \quad n = M, M+1, M+2, \dots \quad (5.11.6)$$

About these formulae see also (5.13.22), (5.13.23).

We see that when  $r$  has zeroes, then the element of the Grassmannian defining the solution corresponds to the finite-dimensional Grassmannian.

If  $r(0) = 0$  one can construct a plane

$$\{w_\infty^*(n, \mathbf{0}, \mathbf{t}^*, 1/z), n = 1, \dots, M\} \quad (5.11.7)$$

which is spanned by a finite number of vectors. The space spanned by these vectors and in addition by the vectors (5.11.5) gives the half infinite plane generated by  $\{z^n, n \geq 0\}$ .

## 5.12 Rational $r$ and their $q$ -deformations

**Lemma 1** If

$$r(n) = \frac{\prod_{i=1}^p (n + a_i)}{\prod_{i=1}^s (n + b_i)} \quad (5.12.1)$$

then

$$r_\lambda(n) = (s_\lambda(\gamma(+\infty)))^{s-p} \frac{\prod_{k=1}^p s_\lambda(\gamma(a_k + n))}{\prod_{k=1}^s s_\lambda(\gamma(b_k + n))} \quad (5.12.2)$$

where

$$\gamma(\infty) = (1, 0, 0, \dots), \quad \gamma(a) = \left(\frac{a}{1}, \frac{a}{2}, \frac{a}{3}, \dots\right) \quad (5.12.3)$$

and  $s_\lambda(\gamma)$  are Schur functions as functions of variables  $\gamma$  according to (2.1.9) and (2.1.12).

**Lemma 2** If

$$r(n) = \frac{\prod_{i=1}^p (1 + q^{n+a_i})}{\prod_{i=1}^s (1 + q^{n+b_i})} \quad (5.12.4)$$

then

$$r_\lambda(n) = (s_\lambda(\gamma(+\infty, q)))^{s-p} \frac{\prod_{k=1}^p s_\lambda(\gamma(a_k + n, q))}{\prod_{k=1}^s s_\lambda(\gamma(b_k + n, q))} \quad (5.12.5)$$

where

$$\gamma_m(a, q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad \gamma(+\infty, q) = \frac{1}{m(1 - q^m)}, \quad m = 1, 2, \dots \quad (5.12.6)$$

For the proof we use the following relations (see [1]):

$$(q^a; q)_\lambda = \prod_{(i,j) \in \lambda} (1 - q^{a+j-i}) = \frac{s_\lambda(\gamma(a, q))}{s_\lambda(\gamma(+\infty, q))}, \quad (a)_\lambda = \prod_{(i,j) \in \lambda} (a + j - i) = \frac{s_\lambda(\gamma(a))}{s_\lambda(\gamma(+\infty))}, \quad (5.12.7)$$

Now we see that for arbitrary complex  $b$  and for  $r$  of (5.12.1) we have

$$\langle e^{\gamma_1}, e^{\sum_{m=1}^{\infty} m \gamma_m t_m^*} \rangle_{r,n} = \langle e^{(b+n) \sum_{m=1}^{\infty} m \gamma_m}, e^{\sum_{m=1}^{\infty} m \gamma_m t_m^*} \rangle_{r_b, n}, \quad r_b(n) = \frac{r(n)}{b+n}. \quad (5.12.8)$$

For  $r$  of (5.12.4) we have

$$\langle e^{\sum_{m=1}^{\infty} \frac{\gamma_m}{1-q^m}}, e^{\sum_{m=1}^{\infty} m\gamma_m t_m^*} \rangle_{r,n} = \langle e^{\sum_{m=1}^{\infty} \frac{\gamma_m(1-q^{b+n+m})}{1-q^m}}, e^{\sum_{m=1}^{\infty} m\gamma_m t_m^*} \rangle_{r_b,n}, \quad r_b(n) = \frac{r(n)}{1-q^{b+n}}. \quad (5.12.9)$$

Let us note that parameters  $\gamma_m(a, q)$  and  $\gamma_m(a)$  are chosen via 'generalized Hirota-Miwa transforms' (5.12.6), (5.12.3) with 'multiplicity  $a$ '. Remember that  $|q| < 1$ . Due to

$$s_\lambda(\gamma(+\infty, q)) = \lim_{a \rightarrow +\infty} s_\lambda(\gamma(a, q)) = \frac{q^{n(\lambda)}}{H_\lambda(q)}, \quad (5.12.10)$$

$$s_\lambda(\gamma(+\infty)) = \lim_{a \rightarrow +\infty} s_\lambda\left(\frac{\gamma_1(a)}{a}, \frac{\gamma_2(a)}{a^2}, \dots\right) = \lim_{a \rightarrow +\infty} \frac{1}{a^{|\lambda|}} s_\lambda(\gamma(a)) = \frac{1}{H_\lambda}. \quad (5.12.11)$$

This allows to rewrite the series (5.10.7) and (5.10.24) only in terms of Schur functions, see [9].

For functions  ${}_p F_s$  of (5.10.7), (5.10.9) and  ${}_p \Phi_s$  of (5.10.24), (5.10.25) we use the results of subsection 5.4 to derive the following representations

$${}_p F_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}^N \right) = \frac{1}{\Delta(x)} e^{x_1 r(D_1) + \dots + x_N r(D_N)} \cdot \Delta(x) \quad (5.12.12)$$

$$= \frac{1}{\Delta(x)} \prod_{i=1}^N \left( 1 - x_i \frac{r(D_i)}{b + D_i} \right)^{-b} \cdot \Delta(x) \quad (5.12.13)$$

$${}_p \Phi_s \left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}^N \right) = \frac{1}{\Delta(x)} \prod_{i=1}^N \frac{1}{(x_i r(D_i); q)_\infty} \cdot \Delta(x) \quad (5.12.14)$$

$$= \frac{1}{\Delta(x)} \prod_{i=1}^N \frac{(x_i q^{b r(D_i)} (1 - q^{b+D_i})^{-1}; q)_\infty}{(x_i r(D_i); q)_\infty} \cdot \Delta(x) \quad (5.12.15)$$

where  $D_i = x_i \frac{\partial}{\partial x_i}$ ,  $\Delta(x)$  is defined in (5.4.8), the notation  $(b; q)_\infty$  see in (7.0.13).

### 5.13 Gauss factorization problem, additional symmetries, string equations and $\Psi DO$ on the circle [9]

Let us describe relevant string equations following Takasaki and Takebe [27], [29]. We shall also consider this topic in a more detailed paper.

Let us introduce infinite matrices to describe KP and TL flows and symmetries, see [23]. Zakharov-Shabat dressing matrices are  $K$  and  $\bar{K}$ .  $K$  is a lower triangular matrix with unit main diagonal:  $(K)_{ii} = 1$ .  $\bar{K}$  is an upper triangular matrix. The matrices  $K, \bar{K}$  depend on parameters  $M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*$ . The matrices  $(\Lambda)_{ik} = \delta_{i,k-1}$ ,  $(\bar{\Lambda})_{ik} = \delta_{i,k+1}$ . For each value of  $\mathbf{t}, \mathbf{T}, \mathbf{t}^*$  and  $M \in \mathbb{Z}$  they solve Gauss (Riemann-Hilbert) factorization problem for infinite matrices:

$$\bar{K} = KG(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*), \quad G(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = \exp(\xi(\mathbf{t}, \Lambda)) \Lambda^M G(\mathbf{0}, \mathbf{T}, \mathbf{0}) \bar{\Lambda}^M \exp(\xi(\mathbf{t}^*, \bar{\Lambda})) \quad (5.13.1)$$

We put  $\log(\bar{K}_{ii}) = \phi_{i+M}$ , and a set of fields  $\phi_i(\mathbf{t}, \mathbf{t}^*)$ ,  $(-\infty < i < +\infty)$  solves the hierarchy of higher two-dimensional TL equations.

Take  $L = K\Lambda K^{-1}$ ,  $\bar{L} = \bar{K}\bar{\Lambda}\bar{K}^{-1}$ , and  $(\Delta)_{ik} = i\delta_{i,k}$ ,  $\widehat{M} = K\Delta K^{-1} + M + \sum n t_n L^n$ ,  $\widehat{\bar{M}} = \bar{K}\Delta\bar{K}^{-1} + M + \sum n t_n^* \bar{L}^n$ . Then the KP additional symmetries [27], [29], [30], [10], [31] and higher TL flows [23] are written as

$$\partial_{\mathbf{t}_n^*} K = - \left( (r(\widehat{M}) L^{-1})^n \right)_- K, \quad \partial_{\mathbf{t}_n} \bar{K} = \left( (r(\widehat{M}) L^{-1})^n \right)_+ \bar{K}, \quad (5.13.2)$$



$$\partial_{t_n^*} K = -(\bar{L}^n)_- K, \quad \partial_{t_n^*} \bar{K} = (\bar{L}^n)_+ \bar{K}. \quad (5.13.3)$$

Then the string equations are

$$\bar{L}L = r(\widehat{M}), \quad (5.13.4)$$

$$\widehat{\bar{M}} = \widehat{M}. \quad (5.13.5)$$

The first equation is a manifestation of the fact that the group time  $\mathbf{t}_1^*$  of the additional symmetry of KP can be identified with the Toda lattice time  $t_1^*$ . In terms of tau-function we have the equation (5.5.6) in terms of vertex operator action [30],[10], or the equations (5.4.19) in case the tau-function is written in Hirota-Miwa variables.

The second string equation (5.13.5) is related to the symmetry of our tau-functions with respect to  $\mathbf{t} \leftrightarrow \mathbf{t}^*$ .

When

$$r(M) = M + a, \quad (5.13.6)$$

the equations (5.13.4),(5.13.5) describe  $c = 1$  string, see [28],[29]. In this case we easily get the relation

$$[\bar{L}, L] = 1. \quad (5.13.7)$$

The string equations in the form of Takasaki allows us to notice the similarity to the different problem. The dispersionless limit of (5.13.7) (and also of (5.13.5), (5.13.4), where  $r(M) = M^n$ , and of (5.13.6)) will be written as

$$\bar{\lambda}\lambda = \mu^n, \quad n \in \mathbb{Z}, \quad (5.13.8)$$

$$\bar{\mu} = \mu. \quad (5.13.9)$$

The case when  $\bar{\lambda}$  and  $\bar{\mu}$  are complex conjugate of  $\lambda$  and of  $\mu$  respectively, is of interest. These string equations (mainly the case  $n = 1$ ) were recently investigated to solve the so-called Laplacian growth problem, see [38]. We are grateful to A.Zabrodin for the discussion on this problem. For the dispersionless limit of the KP and TL hierarchies see [27].

In case the function  $r(n)$  has zeroes (described by divisor  $\mathbf{m} = (M_1, \dots)$ :  $r(M_k) = 0$ ), one needs to produce the replacement:

$$\Lambda \rightarrow \Lambda(\mathbf{m}), \quad \bar{\Lambda} \rightarrow \bar{\Lambda}(\mathbf{m}), \quad (5.13.10)$$

where new matrices  $\Lambda(\mathbf{m}), \bar{\Lambda}(\mathbf{m})$  are defined as

$$(\Lambda(\mathbf{m}))_{i,j} = \delta_{i,j-1}, j \neq M_k, \quad (\Lambda(\mathbf{m}))_{i,j} = 0, j = M_k, \quad (5.13.11)$$

$$(\bar{\Lambda}(\mathbf{m}))_{i,j} = \delta_{i,j+1}, i \neq M_k, \quad (\bar{\Lambda}(\mathbf{m}))_{i,j} = 0, i = M_k. \quad (5.13.12)$$

This modification describes the open TL equation (??):

$$\partial_{t_1} \partial_{t_1^*} \varphi_n = \delta(n) e^{\varphi_{n-1} - \varphi_n} - \delta(n+1) e^{\varphi_n - \varphi_{n+1}}. \quad (5.13.13)$$

The set of fields  $\phi_\infty, \dots, \phi_{M_1}, \phi_{M_1+1} \dots$  consists of the following parts due to the conditions

$$\sum_{n=-\infty}^{M_s} \phi_n = 0, \quad \sum_{n=M_k+1}^{M_{k+1}} \phi_n = 0, \quad \sum_{n=\infty}^{M_1+1} \phi_n = 0, \quad (5.13.14)$$

which result from  $\tau(M_k, \mathbf{t}, \mathbf{t}^*) = 1$ .

**Remark 2** The matrix  $r(\widehat{M})$  contains  $(\widehat{M} - b_i)$  in the denominator. The matrix  $(\widehat{M} - b_i)^{-1}$  is  $K(\Delta - b_i)^{-1}K^{-1}(1 + O(\mathbf{t}))$  (compare the consideration of the inverse operators with [31]).

The KP tau-function (3.4.8) can be obtained as follows.

$$G(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = G(\mathbf{0}, \mathbf{T}, \mathbf{0})U(M, \mathbf{t}, \mathbf{t}^*), \quad U(M, \mathbf{t}, \mathbf{t}^*) = U^+(\mathbf{t})U^-(M, \mathbf{t}^*). \quad (5.13.15)$$

$$U^+(\mathbf{t}) = \exp(\xi(\mathbf{t}, \Lambda)), \quad U^-(M, \mathbf{t}^*) = \exp(\xi(\mathbf{t}^*, \bar{\Lambda}r(\Delta + M))), \quad (5.13.16)$$

The matrix  $G(\mathbf{0}, \mathbf{T}, \mathbf{0})$  is related to the transformation of the eq.(5.3.14) to the eq.(5.3.11). By taking the projection [23]  $U \mapsto U_{--}$  for non positive values of matrix indices we obtain a determinant representation of the tau-function (3.4.8):

$$\tau_r(M, \mathbf{t}, \mathbf{t}^*) = \frac{\det U_{--}(M, \mathbf{t}, \mathbf{t}^*)}{\det(U_{--}^+(\mathbf{t})) \det(U_{--}^-(M, \mathbf{t}^*))} = \det U_{--}(M, \mathbf{t}, \mathbf{t}^*), \quad (5.13.17)$$

since both determinants in the denominator are equal to one. Formula (5.13.17) is also a Segal-Wilson formula for  $GL(\infty)$  2-cocycle [39]  $C_M(U^+(-\mathbf{t}), U^-(-\mathbf{t}^*))$ . Choosing the function  $r$  as in Section 3.2 we obtain hypergeometric functions listed in the *Introduction*.

**Remark 3** Therefore the hypergeometric functions which were considered above have the meaning of  $GL(\infty)$  two-cocycle on the two multi-parametric group elements  $U^+(\mathbf{t})$  and  $U^-(M, \mathbf{t}^*)$ . Both elements  $U^+(\mathbf{t})$  and  $U^-(M, \mathbf{t}^*)$  can be considered as elements of group of pseudo-differential operators on the circle. The corresponding Lie algebras consist of the multiplication operators  $\{z^n; n \in N_0\}$  and of the pseudo-differential operators  $\left\{\left(\frac{1}{z}r\left(z\frac{d}{dz} + M\right)\right)^n; n \in N_0\right\}$ . Two sets of group times  $\mathbf{t}$  and  $\mathbf{t}^*$  play the role of indeterminates of the hypergeometric functions (5.10.2). Formulas (3.4.8) and (5.3.8) mean the expansion of  $GL(\infty)$  group 2-cocycle in terms of corresponding Lie algebra 2-cocycle

$$c_M(z, \frac{1}{z}r(D)) = r(M), \quad c_M(\tilde{r}(D)z, \frac{1}{z}r(D + M)) = \tilde{r}(M)r(M). \quad (5.13.18)$$

Japanese cocycle is cohomological to Khesin-Kravchenko cocycle [44] for the  $\Psi DO$  on the circle:

$$c_M \sim c_0 \sim \omega_M, \quad (5.13.19)$$

which is

$$\omega_M(A, B) = \frac{1}{2\pi\sqrt{-1}} \oint \text{res}_{\partial} A[\log(D + M), B]dz, \quad A, B \in \Psi DO. \quad (5.13.20)$$

For the group cocycle we have

$$C_M\left(e^{-\sum z^n t_n}, e^{-\sum (z^{-1}r(D))^n \mathbf{t}_n^*}\right) = \tau_r(M, \mathbf{t}, \mathbf{t}^*), \quad (5.13.21)$$

where we imply that the order of  $\Psi DO$   $r(D)$  is 1 or less. About properties of  $e^{-\sum (z^{-1}r(D))^n \mathbf{t}_n^*}$  see (??), (??).

**Remark 4** It is interesting to note that in case of hypergeometric functions  ${}_pF_s$  (7.0.16) the order of  $r$  is  $p - s$  (see Example 3), and the condition  $p - s \leq 1$  is the condition of the convergence of this hypergeometric series, see [21]. Namely the radius of convergence is finite in case  $p - s = 1$ , it is infinite when  $p - s < 1$  and it is zero for  $p - s > 1$  (this is true for the case when no one of  $a_k$  in (7.0.16) is nonnegative integer).

**Remark 5** The set of functions  $\{w(n, z), n = M, M + 1, M + 2, \dots\}$ , where

$$w(n, z) = \exp \left( - \sum_{m=0}^{\infty} t_m^* \left( \frac{1}{z} r(D) \right)^m \right) \cdot z^n, \quad (5.13.22)$$

may be identified with Sato Grassmannian (5.11.5) related to the cocycle (5.13.21). The dual Grassmannian (5.11.6) is the set of one forms  $\{w^*(n, z), n = M, M + 1, M + 2, \dots\}$ ,

$$w^*(n, z) = \exp \left( \sum_{m=0}^{\infty} t_m^* \left( \frac{1}{z} r(-D) \right)^m \right) \cdot z^{-n} dz. \quad (5.13.23)$$

## 6 Appendices B

### 6.1 A scalar product in the theory of symmetric functions [1]

We consider a ring of symmetric functions of variables  $x_i, i = 1, 2, \dots, N$ , where  $N$  may be infinity. Let us remember the notion of scalar product. Different choices of scalar product give rise to different systems of bi-orthogonal symmetric polynomial functions (like Schur functions, Macdonald functions and so on).

Actually we consider two sets of variables, the first one is  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ , the second is  $\mathbf{p} = (p_1, p_2, \dots)$ . These two sets are related via

$$p_m = \sum_{i=1}^N x_i^m, \quad m = 1, 2, \dots \quad (6.1.1)$$

If one takes

$$e \sum_{m=1}^{\infty} \frac{1}{m} v_m^{-1} p_m p_m^* = \sum_{\lambda} P_{\lambda}(\mathbf{p}) Q_{\lambda}(\mathbf{p}^*) = \sum_{\lambda} \frac{p_{\lambda} p_{\lambda}^*}{z_{\lambda}^{(v)}}, \quad (6.1.2)$$

then for any partitions  $\mu, \lambda$

$$\langle P_{\mu}, Q_{\lambda} \rangle_v = \delta_{\mu, \lambda} \quad (6.1.3)$$

with respect to the scalar product

$$\langle p_{\mu}, p_{\lambda} \rangle_v = z_{\lambda}^{(v)} \delta_{\mu, \lambda} \quad (6.1.4)$$

When  $v = 1$  then

$$z_{\lambda}^{(v)} = z_{\lambda}, \quad z_{\lambda} = \prod_i i^{m_i} m_i! \quad (6.1.5)$$

where  $m_i = m_i(\lambda)$  is the number of parts of  $\lambda$  equal to  $i$

$$p_{\mu} = \prod_i p_{\mu_i}, \quad p_{\mu_i} = \sum_k x_k^{\mu_i}, \quad m t_m = p_m \quad (6.1.6)$$

If one takes

$$v_m = \frac{1 - t^m}{1 - q^m} \quad (6.1.7)$$

he gets Macdonald's polynomials  $Q_{\lambda}(\mathbf{p}) = Q_{\lambda}(\mathbf{p}; q, t)$ ,  $P_{\lambda}(\mathbf{p}) = P_{\lambda}(\mathbf{p}; q, t)$ . In this case

$$z_{\lambda}^{(v)} = z_{\lambda}(q, t) = z_{\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad z_{\lambda} = \prod_i i^{m_i} m_i! \quad (6.1.8)$$

where  $l(\lambda)$  is the length of  $\lambda$  and  $m_i = m_i(\lambda)$  is the number of parts of  $\lambda$  equal to  $i$ .

## 6.2 A deformation of the scalar product and series of hypergeometric type

For different choice of function  $v$  we choose numbers  $a^{(v)}, b^{(v)}$  and a function  $r^{(v)}$  which is a function of one variable  $M$ , say. We define

$$r_\lambda^{(v)}(M) = \prod_{i,j \in \lambda} r^{(v)}(M + a^{(v)}i + b^{(v)}j) \quad (6.2.1)$$

as a product over all boxes of the Young diagram. In what follows we shall omit the superscript  $(v)$ .

Now let us deform the scalar product (6.1.3) in a following way

$$\langle P_\mu, Q_\lambda \rangle_{v,r,M} = r_\lambda(M) \delta_{\mu,\lambda} \quad (6.2.2)$$

To get series of hypergeometric type we consider

$$\langle e^{\sum_{m=1}^{\infty} \sum_{k=1}^N v_m^{-1} x_k^m \frac{pm}{m}}, e^{\sum_{m=1}^{\infty} \sum_{k=1}^N v_m^{-1} y_k^m \frac{pm}{m}} \rangle_{v,r,M} \quad (6.2.3)$$

Using (6.1.2)

$$\langle e^{\sum_{m=1}^{\infty} \sum_{k=1}^N v_m^{-1} x_k^m \frac{pm}{m}}, e^{\sum_{m=1}^{\infty} \sum_{k=1}^N v_m^{-1} y_k^m \frac{pm}{m}} \rangle_{v,r,M} = \sum_{\lambda} r_\lambda(M) P_\lambda(\mathbf{x}^N) Q_\lambda(\mathbf{y}^N) \quad (6.2.4)$$

In this paper we shall consider two cases of (6.1.7). First one is

$$v_n = 1 \quad (6.2.5)$$

for all  $n$ . The second one is

$$v_{2n} = 0, \quad v_{2n+1} = 2 \quad (6.2.6)$$

The first case corresponds to Schur function series. The second one corresponds to projective Schur function series. These are the cases when the scalar product (6.2.2) can be presented as a vacuum expectation of certain fermionic fields, see *Section 4* and [45].

## 6.3 Scalar product of tau functions. A conjecture.

Before we considered tau functions of hypergeometric type (3.4.8). In this subsection we shall consider more general TL tau functions. Let  $\tau_1(\mathbf{t}, \mathbf{t}^*), \tau_2(\mathbf{t}, \mathbf{t}^*)$  be TL tau functions of the form

$$\tau_1(\mathbf{t}, \mathbf{t}^*) = e^{X(\mathbf{t})} \cdot e^{\sum_{m=1}^{\infty} m t_m t_m^*}, \quad \tau_2(\mathbf{t}, \mathbf{t}^*) = e^{Y(\mathbf{t}^*)} \cdot e^{\sum_{m=1}^{\infty} m t_m t_m^*} \quad (6.3.1)$$

where the operators  $X(\mathbf{t})$  and  $Y(\mathbf{t}^*)$  are linear combinations of vertex operators (5.1.1) and (5.1.2) respectively:

$$X_i(\mathbf{t}) = \int f(z, z') V_\infty(\mathbf{t}, z) V_\infty^*(\mathbf{t}, z') dz dz' \quad (6.3.2)$$

$$Y_i(\mathbf{t}^*) = \int g(z, z') V_0(\mathbf{t}^*, z) V_\infty^*(\mathbf{t}^*, z') dz dz' \quad (6.3.3)$$

If the TL tau functions (6.3.1) have the following forms [24],[25],[26]

$$\tau_1(\mathbf{t}, \mathbf{t}^*) = \sum_{\lambda, \mu} K_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\mathbf{t}^*), \quad \tau_2(\mathbf{t}, \mathbf{t}^*) = \sum_{\lambda, \mu} M_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\mathbf{t}^*) \quad (6.3.4)$$

then their (standard) scalar product (2.1.15) is obviously equal to

$$\langle \tau_1(\mathbf{t}, \gamma), \tau_2(\gamma, \mathbf{t}^*) \rangle = \sum_{\lambda, \mu} N_{\lambda\mu} s_\lambda(\mathbf{t}) s_\mu(\mathbf{t}^*), \quad N_{\lambda\mu} = \sum_{\nu} K_{\lambda\nu} M_{\nu\mu} \quad (6.3.5)$$

It is not immediately clear that what we got is a tau function again (without solving Hirota equations). However on the other hand we have

$$\langle \tau_1(\mathbf{t}, \gamma), \tau_2(\gamma, \mathbf{t}^*) \rangle = e^{X(\mathbf{t})} \cdot e^{Y(\mathbf{t}^*)} \cdot \langle e^{\sum_{m=1}^{\infty} m t_m \gamma_m}, e^{\sum_{m=1}^{\infty} m \gamma_m t_m^*} \rangle = e^{X(\mathbf{t})} \cdot e^{Y(\mathbf{t}^*)} \cdot e^{\sum_{m=1}^{\infty} m t_m t_m^*} \quad (6.3.6)$$

The last relation shows that we get tau function again according to general results of [4]. (It was shown there that the action of vertex operators  $e^X$  to KP tau function gives rise to a new KP tau function. TL tau function is the tau function of the pair of KP hierarchies related to the sets of times  $\mathbf{t}$  and  $\mathbf{t}^*$ , see [23]).

Due to (4.1.1) it means that if the following integral over complex planes  $\gamma_m, m = 1, 2, \dots$

$$\int \tau_1(\mathbf{t}, \gamma) e^{-\sum_{m=1}^{\infty} m |\gamma_m|^2} \tau_2(-\bar{\gamma}, \mathbf{t}^*) \prod_{m=1}^{\infty} \frac{m d^2 \gamma_m}{\pi} \quad (6.3.7)$$

exists, then it is a new TL tau function. Here the variables  $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2, \dots)$  are complex conjugated to  $\gamma = (\gamma_1, \gamma_2, \dots)$ .

The proof can be done also directly as follows

$$\int \tau_1(\mathbf{t}, \gamma) e^{-\sum_{m=1}^{\infty} m |\gamma_m|^2} \tau_2(-\bar{\gamma}, \mathbf{t}^*) \prod_{m=1}^{\infty} \frac{m d^2 \gamma_m}{\pi} = \quad (6.3.8)$$

$$e^{X(\mathbf{t})} \cdot e^{Y(\mathbf{t}^*)} \cdot \int e^{\sum_{m=1}^{\infty} m t_m \gamma_m - m |\gamma_m|^2 - m \bar{\gamma}_m t_m^*} \prod_{m=1}^{\infty} \frac{m d^2 \gamma_m}{\pi} = \quad (6.3.9)$$

$$e^{X(\mathbf{t})} \cdot e^{Y(\mathbf{t}^*)} \cdot e^{\sum_{m=1}^{\infty} m t_m t_m^*} \quad (6.3.10)$$

which is tau function.

Let us try to consider more general situation. Let  $\tau_0(\mathbf{t}, \mathbf{t}^*)$  be a TL tau function with the following properties. First,  $\tau_0(\mathbf{t}, \mathbf{t}^*) = \tau_0(\mathbf{t}^*, \mathbf{t})$ . Second, each integral over complex plane  $\gamma_m, m = 1, 2, \dots$

$$\int \tau_0(\gamma, -\bar{\gamma}) d^2 \gamma_m = a_m \quad (6.3.11)$$

where  $\bar{\gamma}_m$  is complex conjugated of  $\gamma_m$ , is convergent and non-vanishing. Then

$$\int \tau_0(\gamma, -\bar{\gamma}) \prod_{m=1}^{\infty} \frac{d^2 \gamma_m}{a_m} = 1 \quad (6.3.12)$$

Given  $\tau_0$ , we introduce the following scalar product

$$\langle f, g \rangle_0 = \int f(\gamma) \tau_0(\gamma, -\bar{\gamma}) g(-\bar{\gamma}) \prod_{m=1}^{\infty} \frac{d^2 \gamma_m}{a_m} \quad (6.3.13)$$

Let us present a conjecture that scalar product of TL tau functions is TL tau function again.

**Conjecture.** Let  $\tau_1(\mathbf{t}, \mathbf{t}^*), \tau_2(\mathbf{t}, \mathbf{t}^*)$  be TL tau functions. If the integral over complex planes  $\gamma_m, m = 1, 2, \dots$

$$\int \tau_1(\mathbf{t}, \gamma) \tau_0(\gamma, -\bar{\gamma}) \tau_2(-\bar{\gamma}, \mathbf{t}^*) \prod_{m=1}^{\infty} \frac{d^2 \gamma_m}{a_m}, \quad (6.3.14)$$

exists, then it is a new TL tau function.

## 6.4 Integral representation of scalar products

For the scalar product (6.2.2) one can present the following integral realization.

In case  $r = 1$

$$\langle f, g \rangle = \int \cdots \int f(\mathbf{t}) g(\bar{\mathbf{t}}) e^{-\sum_{m=1}^{\infty} m |t_m|^2} \prod_{m=1}^{\infty} \frac{d^2 t_m}{a_m} \quad (6.4.1)$$

where  $mt_m$  are power sum functions,  $m\bar{t}_m$  are complex conjugated variables, and integration is going over complex planes  $(t_m, \bar{t}_m)$ . Normalization factors  $a_m$  depend on  $m$ :

$$a_m = \frac{\pi}{m} \quad (6.4.2)$$

In case  $r(n) = n$  there is a representation

$$\langle f, g \rangle_{r,n} = \int \cdots \int f(\mathbf{z}) g(\bar{\mathbf{z}}) e^{-\sum_{k=1}^N |z_k|^2} |\Delta(z)|^2 \prod_{k=1}^N \frac{d^2 z_k}{b_k} \quad (6.4.3)$$

where  $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_N)$  are complex conjugated to  $\mathbf{z} = (z_1, \dots, z_N)$ , normalizing constants are

$$b_k = \pi \quad (6.4.4)$$

$\Delta$  is Vandermond determinant

$$\Delta(z) = \prod_{i < k}^N (z_k - z_i) \quad (6.4.5)$$

and one puts  $\Delta(z) = 1$  for  $N = 1$ . The integration is going over complex planes  $(z_k, \bar{z}_k)$ .

For the scalar product (6.1.4) we present the representation

$$\langle f, g \rangle_v = \int \cdots \int f(\mathbf{t}) g(\bar{\mathbf{t}}) e^{-\sum_{m=1}^{\infty} \frac{m}{v_m} t_m \bar{t}_m} \prod_{m=1}^{\infty} \frac{dt_m d\bar{t}_m}{a_m} \quad (6.4.6)$$

where  $mt_m$  are power sum functions,  $m\bar{t}_m$  are complex conjugated variables, and integration is going over complex planes  $(t_m, \bar{t}_m)$ . Normalization factors  $a_m$  depend on  $m$ :

$$a_m = \frac{2\pi v_m}{m} \sqrt{-1} \quad (6.4.7)$$

In particular for Macdonald's polynomials  $Q_\lambda(\mathbf{p}; q, t)$ ,  $P_\lambda(\mathbf{p}; q, t)$ , see (6.1.7), we get

$$\int \cdots \int Q_\lambda(\mathbf{p}; q, t) P_\lambda(\bar{\mathbf{p}}; q, t) e^{-\sum_{m=1}^{\infty} \mathbf{p}_m \bar{\mathbf{p}}_m \frac{1-q^m}{m(1-t^m)}} \prod_{m=1}^{\infty} \frac{dp_m d\bar{p}_m}{a_m} = \delta_{\mu, \lambda} \quad (6.4.8)$$

$$a_m = 2\pi \frac{1 - q^m}{m(1 - t^m)} \sqrt{-1} \quad (6.4.9)$$

## 7 Appendix C. Hypergeometric functions

**Ordinary hypergeometric functions** First let us remember that generalized hypergeometric function of one variable  $x$  is defined as

$${}_pF_s(a_1, \dots, a_p; b_1, \dots, b_s; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_s)_n} \frac{x^n}{n!}. \quad (7.0.10)$$

Here  $(a)_n$  is Pochhammer's symbol:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \cdots (a+n-1). \quad (7.0.11)$$

Given number  $q$ ,  $|q| < 1$ , the so-called basic hypergeometric series of one variable is defined as

$${}_p\Phi_s(a_1, \dots, a_p; b_1, \dots, b_s; q, x) = \sum_{n=0}^{\infty} \frac{(q^{a_1}; q)_n \cdots (q^{a_p}; q)_n}{(q^{b_1}; q)_n \cdots (q^{b_s}; q)_n} \frac{x^n}{(q; q)_n}. \quad (7.0.12)$$

Here  $(q^a, q)_n$  is  $q$ -deformed Pochhammer's symbol:

$$(b; q)_0 = 1, \quad (b; q)_n = (1-b)(1-bq) \cdots (1-bq^{n-1}). \quad (7.0.13)$$

Both series converge for all  $x$  in case  $p < s+1$ . In case  $p = s+1$  they converge for  $|x| < 1$ . We refer these well-known hypergeometric functions as ordinary hypergeometric functions.

**The multiple hypergeometric series related to Schur polynomials [22],[21],[34].** There are several well-known different multi-variable generalizations of hypergeometric series of one variable [22, 21]. If one replaces the sum over  $n$  to a sum over partitions  $\lambda = (n_1, n_2, \dots)$ , and replaces the single variable  $x$  to a Hermitian matrix  $X$ , he get one of generalizations of hypergeometric series which is called hypergeometric function of matrix argument  $\mathbf{X}$  with indices  $\mathbf{a}$  and  $\mathbf{b}$  [35],[22]:

$${}_pF_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| \mathbf{X} \right) = \sum_{l(\lambda) \leq N} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} \frac{Z_\lambda(\mathbf{X})}{|\lambda|!}. \quad (7.0.14)$$

Here the sum is over all different partitions  $\lambda = (n_1, n_2, \dots, n_k)$ , where  $n_1 \geq n_2 \geq \cdots \geq n_k$ ,  $k \leq |\lambda|$ ,  $|\lambda| = n_1 + \cdots + n_r$  and whose length  $l(\lambda) = k \leq N$  ( $n_k \neq 0$ ).  $\mathbf{X}$  is a Hermitian  $N \times N$  matrix, and  $Z_\lambda(\mathbf{X})$  are zonal spherical polynomial for the symmetric spaces of the following types:  $GL(N, R)/SO(N)$ ,  $GL(N, C)/U(N)$ , or  $GL(N, H)/Sp(N)$  see [21]. The definition of symbol  $(a)_\lambda$  depends on the choice of the symmetric space:

$$(a)_\lambda = (a)_{n_1} (a - \frac{1}{\alpha})_{n_2} \cdots (a - \frac{k-1}{\alpha})_{n_k}, \quad (a)_0 = 1, \quad (7.0.15)$$

where  $\alpha = 2$ ,  $\alpha = 1$  and  $\alpha = \frac{1}{2}$  for the symmetric spaces  $GL(N, R)/SO(N)$ ,  $GL(N, C)/U(N)$ , and  $GL(N, H)/Sp(N)$  respectively. The function (7.0.14) actually depends on the eigenvalues of matrix  $X$  which are  $\mathbf{x}^N = (x_1, \dots, x_N)$ . In what follows we consider only the case of  $GL(N, C)/U(N)$  symmetric space. Then zonal spherical polynomial  $Z_\lambda(\mathbf{X})$  is proportional to the Schur function  $s_\lambda(x_1, x_2, \dots, x_N)$  corresponding to a partition  $\lambda$  [1],  $s_\lambda(x_1, x_2, \dots, x_N)$  is a symmetric function of variables  $x_k$ .

For this choice of the symmetric space the hypergeometric function can be rewritten as follows

$${}_pF_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| \mathbf{X} \right) = \sum_{l(\lambda) \leq N} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} \frac{s_\lambda(\mathbf{x}^N)}{H_\lambda}. \quad (7.0.16)$$

where  $H_\lambda$  is 'hook product':

$$H_\lambda = \prod_{(i,j) \in \lambda} h_{ij}, \quad h_{ij} = (n_i + n'_j - i - j + 1) \quad (7.0.17)$$

and

$$(a)_\lambda = (a)_{n_1}(a-1)_{n_2} \cdots (a-k+1)_{n_k}, \quad (a)_0 = 1 \quad (7.0.18)$$

Taking  $N = 1$  we get (7.0.10).

For this hypergeometric functions we suggested different representations like (5.12.12),(5.12.13) or like (5.10.14),(5.10.15),(5.10.16).

Let  $|q| < 1$ . The multiple *basic* hypergeometric series related to Schur polynomials were suggested by L.G.Macdonald and studied by S.Milne [34]. These series are as follows

$${}_p\Phi_s(a_1, \dots, a_p; b_1, \dots, b_s; q, \mathbf{x}^N) = \sum_{\lambda} \frac{(q^{a_1}; q)_\lambda \cdots (q^{a_p}; q)_\lambda}{(q^{b_1}; q)_\lambda \cdots (q^{b_s}; q)_\lambda} \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(\mathbf{x}^N), \quad (7.0.19)$$

where the sum is over all different partitions  $\lambda = (n_1, n_2, \dots, n_k)$ , where  $n_1 \geq n_2 \geq \dots \geq n_k \geq 0$ ,  $k \leq |\lambda|$ ,  $|\lambda| = n_1 + \dots + n_k$  and whose length  $l(\lambda) = k \leq N$ . Schur polynomial  $s_\lambda(\mathbf{x}^N)$ , with  $N \geq l(\lambda)$ , is a symmetric function of variables  $\mathbf{x}^N$  and defined as follows [1]:

$$s_\lambda(\mathbf{x}^N) = \frac{a_{\lambda+\delta}}{a_\delta}, \quad a_\lambda = \det(x_i^{n_j})_{1 \leq i, j \leq N}, \quad \delta = (N-1, N-2, \dots, 1, 0). \quad (7.0.20)$$

Coefficient  $(q^c; q)_\lambda$  associated with partition  $\lambda$  is expressed in terms of the  $q$ -deformed Pochhammer's Symbols  $(q^c; q)_n$  (7.0.13):

$$(q^c; q)_\lambda = (q^c; q)_{n_1} (q^{c-1}; q)_{n_2} \cdots (q^{c-k+1}; q)_{n_k}. \quad (7.0.21)$$

The multiple  $q^{n(\lambda)}$  defined on the partition  $\lambda$ :

$$q^{n(\lambda)} = q^{\sum_{i=1}^k (i-1)n_i}, \quad (7.0.22)$$

and  $q$ -deformed 'hook polynomial'  $H_\lambda(q)$  is

$$H_\lambda(q) = \prod_{(i,j) \in \lambda} (1 - q^{h_{ij}}), \quad h_{ij} = (n_i + n'_j - i - j + 1), \quad (7.0.23)$$

where  $\lambda'$  is the conjugated partition (for the definition see [1]).

For this hypergeometric functions we suggested different representations like (5.12.14),(5.12.15) or like (5.10.28),(5.10.29).

For  $N = 1$  we get (7.0.12).

Let us note that in the limit  $q \rightarrow 1$  series (7.0.19) reduces to (7.0.16), see [21].

### Hypergeometric series of double set of arguments [22],[21],[34]

Another generalization of hypergeometric series is so-called hypergeometric function of two matrix arguments  $\mathbf{X}, \mathbf{Y}$  with indices  $\mathbf{a}$  and  $\mathbf{b}$ :

$${}_p\mathcal{F}_s(a_1, \dots, a_p; b_1, \dots, b_s; \mathbf{X}, \mathbf{Y}) = \sum_{\lambda} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} \frac{Z_\lambda(\mathbf{X}) Z_\lambda(\mathbf{Y})}{|\lambda|! Z_\lambda(\mathbf{I}_N)}. \quad (7.0.24)$$

Here  $\mathbf{X}, \mathbf{Y}$  are Hermitian  $N \times N$  matrices and  $Z_\lambda(\mathbf{X}), Z_\lambda(\mathbf{Y})$  are zonal spherical polynomials for the symmetric spaces  $GL(N, C)/U(N)$ ,  $GL(N, R)/SO(N)$  and  $GL(N, H)/Sp(N)$  see [21]. The notations are the same as in previous subsection. Again we shall consider only the case of  $GL(N, C)/U(N)$  symmetric spaces. In this case (7.0.24) it may be written as

$${}_p\mathcal{F}_s(a_1, \dots, a_p; b_1, \dots, b_s; \mathbf{x}^{(N)}, \mathbf{y}^{(N)}) = \sum_{\lambda} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_s)_\lambda} \frac{s_\lambda(\mathbf{x}^N) s_\lambda(\mathbf{y}^N)}{H_\lambda s_\lambda(\mathbf{1}^N)}. \quad (7.0.25)$$



We suggest a different representations like (5.10.19) or different integral representations.

The  $q$ -deformation of the hypergeometric function (7.0.25) is as follows

$${}_p\Phi_s \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_s \end{matrix} \middle| q, \mathbf{x}^N, \mathbf{y}^N \right) = \sum_{\substack{\lambda \\ l(\lambda) \leq N}} \frac{(q^{a_1}; q)_\lambda \cdots (q^{a_p}; q)_\lambda}{(q^{b_1}; q)_\lambda \cdots (q^{b_s}; q)_\lambda} \frac{q^{n(\lambda)}}{H_\lambda(q)} \frac{s_\lambda(\mathbf{x}^N) s_\lambda(\mathbf{y}^N)}{s_\lambda(1, q, q^2, \dots, q^{N-1})} \quad (7.0.26)$$

This is the multi-variable basic hypergeometric function of two sets of variables which was also studied by S.Milne, see [34], [21].

There are also hypergeometric functions related to Jack polynomials  $C_\lambda^{(d)}$  [21]:

$${}_p\mathcal{F}_s^{(d)} \left( a_1, \dots, a_p; b_1, \dots, b_s; \mathbf{x}^N, \mathbf{y}^N \right) = \sum_{\lambda} \frac{(a_1)_\lambda^{(d)} \cdots (a_p)_\lambda^{(d)}}{(b_1)_\lambda^{(d)} \cdots (b_s)_\lambda^{(d)}} \frac{C_\lambda^{(d)}(\mathbf{x}^N) C_\lambda^{(d)}(\mathbf{y}^N)}{|\lambda|! C_\lambda^{(d)}(1^n)}, \quad (7.0.27)$$

where

$$(a)_\lambda^{(d)} = \prod_{i=1}^{l(\lambda)} \left( a - \frac{d}{2}(i-1) \right)_{n_i}. \quad (7.0.28)$$

Here  $(c)_k = c(c+1) \cdots (c+k-1)$ . For the special value  $d = 2$  the last expression (7.0.27) coincides with (7.0.19), and reduces to (7.0.25) as  $|q| \rightarrow 1$ . The open problem is to get a fermionic representation of (7.0.27) for arbitrary value of the parameter  $d$ .

## 8 Further generalization. Examples of Gelfand-Graev hypergeometric functions

### 8.1 Scalar product and series in plane partitions

Let us consider the scalar product (3.3.9)

$$\langle s_\mu, s_\lambda \rangle_{r,n} = r_\lambda(n) \delta_{\mu,\lambda} \quad (8.1.1)$$

For the simplicity of notations sometimes we shall write  $\langle, \rangle_r$  instead of  $\langle, \rangle_{r,n}$ .

We introduce notations

$$(\mathbf{t}, \mathbf{p}) = \sum_{m=1}^{\infty} p_m t_m, \quad (\mathbf{p}, \gamma_i) = \sum_{m=1}^{\infty} p_m \gamma_{im}, \quad (\mathbf{p}, \gamma_i^*) = \sum_{m=1}^{\infty} p_m \gamma_{im}^* \quad (8.1.2)$$

**Lemma**

$$\langle s_\lambda, e^{(\mathbf{p}, \gamma)} s_\mu \rangle_{r,n} = \langle s_\mu e^{(\mathbf{p}, \gamma)}, s_\lambda \rangle = r_\lambda(n) s_{\lambda/\mu}(\gamma) \quad (8.1.3)$$

One sees that for  $\gamma = 0$  and  $r = 1$  the right hand side is equal to  $\delta_{\mu,\lambda}$ .

**Proof** One develops  $e^{(\mathbf{p}, \gamma_i)}$  using

$$e^{(\mathbf{p}, \gamma_i)} = \sum_{\nu} s_\nu(\tilde{\mathbf{p}}) s_\nu(\gamma_i) \quad (8.1.4)$$

and taking into account the orthogonality of Schur functions (8.1.1) and the formulas (see section 5 of chapter I of [1])

$$s_\nu s_\mu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda, \quad s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu \quad (8.1.5)$$

Let us consider a set of scalar products (8.1.1) which differ by the choice of  $r$ . The scalar product of functions of  $p_i$  will be labelled by script  $i$ : we shall denote this scalar product as  $\langle, \rangle_{r^{(i)}}$ .

**Lemma**

$$e^{(\mathbf{p}_i, \gamma_i + \tilde{\mathbf{p}}_{i-1})} = \sum_{\nu_1, \nu_2} s_{\nu_1}(\tilde{\mathbf{p}}_i) s_{\nu_1/\nu_2}(\gamma_i) s_{\nu_2}(\tilde{\mathbf{p}}_{i-1}) \quad (8.1.6)$$

and for given partition  $\nu_{i-1}$  we have

$$\langle e^{(\mathbf{p}_i, \gamma_i + \tilde{\mathbf{p}}_{i-1})}, s_{\nu_{i-1}}(\tilde{\mathbf{p}}_{i-1}) \rangle_{r^{(i-1)}, n_{i-1}} = \sum_{\nu_i} s_{\nu_i}(\tilde{\mathbf{p}}_i) s_{\nu_i/\nu_{i-1}}(\gamma_i) r_{\nu_{i-1}}^{(i-1)}(n_{i-1}) \quad (8.1.7)$$

where

$$\tilde{\mathbf{p}}_i = \left( \frac{p_{i1}}{1}, \frac{p_{i2}}{2}, \frac{p_{i3}}{3}, \dots \right) \quad (8.1.8)$$

**Proof** follows from the relations (see [1])

$$e^{\sum_{n=1}^\infty (p_{in} \gamma_{in} + \frac{1}{n} p_{in} p_{i-1, n})} = \sum_{\nu_1} s_{\nu_1}(\gamma_i + \tilde{\mathbf{p}}_{i-1}) s_{\nu_1}(\tilde{\mathbf{p}}_i), \quad s_{\nu_2}(\gamma_i + \tilde{\mathbf{p}}_{i-1}) = \sum_{\nu_2} s_{\nu_1/\nu_2}(\gamma_i) s_{\nu_2}(\tilde{\mathbf{p}}_{i-1}) \quad (8.1.9)$$

and definition (3.3.9).

Let us consider a set of  $\{\mathbf{p}_i, i = 0, 1, 2, \dots, k+1\}$  and a set of  $\{\gamma_i, i = 1, 2, \dots, k+1\}$  with conditions

$$\mathbf{p}_0 = 0, \quad \mathbf{p}_{k+1} = \mathbf{p} \quad (8.1.10)$$

With the help of Lemma we consider scalar product of scalar products (compare with (3.4.7)):

$$\langle e^{(\mathbf{p}_k, \gamma_k + \tilde{\mathbf{p}}_{k-1})} \dots, \langle e^{(\mathbf{p}_3, \gamma_3 + \tilde{\mathbf{p}}_2)}, \langle e^{(\mathbf{p}_2, \gamma_2 + \tilde{\mathbf{p}}_1)}, e^{(\mathbf{p}_1, \gamma_1 + \tilde{\mathbf{p}}_0)} \rangle_{r^{(1)}, n_1} \rangle_{r^{(2)}, n_2} \dots \rangle_{r^{(k-1)}, n_k} \quad (8.1.11)$$

$$= \sum_{\nu_k \geq \dots \geq \nu_0} r_{\nu_{k-1}}^{(k-1)}(n_{k-1}) r_{\nu_{k-2}}^{(k-2)}(n_{k-2}) \dots r_{\nu_2}^{(2)}(n_2) r_{\nu_1}^{(1)}(n_1)$$

$$s_{\nu_k}(\tilde{\mathbf{p}}_k) s_{\nu_k/\nu_{k-1}}(\gamma_k) s_{\nu_{k-1}/\nu_{k-2}}(\gamma_{k-1}) \dots s_{\nu_2/\nu_1}(\gamma_2) s_{\nu_1/\nu_0}(\gamma_1) s_{\nu_0}(\tilde{\mathbf{p}}_0) \quad (8.1.12)$$

$$= I_k(\mathbf{p}, \mathbf{r}^{k-1}, \lambda^{k-1}, \gamma, \mathbf{p}_0) \quad (8.1.13)$$

where  $\mathbf{r}^{k-1} = (r^{(1)}, r^{(2)}, \dots, r^{(k-1)})$ ,  $\lambda_{k-1} = (n_1, \dots, n_{k-1})$ ,  $s_{\nu_0}(\tilde{\mathbf{p}}_0) = \delta_{0, \nu_0}$ .

This is the sum over all *plane partitions* [1] with the largest part  $k$ :  $\pi^{(k)} = (\nu_1 \leq \dots \leq \nu_k)$ .

Now let us consider scalar product of different  $I$  and with respect to standard scalar product (8.1.1). Putting  $\mathbf{p}_0 = 0$  we get a function of  $\gamma = \{\gamma_{in}, i = 1, \dots, k, n = 1, 2, \dots\}$ ,  $\gamma^* = \{\gamma_{in}^*, i = 1, \dots, l, n = 1, 2, \dots\}$ , which will appear to be a tau function

$$\tau_{\tilde{\mathbf{r}}, \mathbf{r}}(\lambda_{l-1}, \lambda_k^*, \gamma, \gamma^*) = \langle I_k(\mathbf{p}, \tilde{\mathbf{r}}_{k-1}, \gamma, 0), I_l(\mathbf{p}, \mathbf{r}_{l-1}, \gamma^*, 0) \rangle_{r^{(k)}, n_k} \quad (8.1.14)$$

where  $\tilde{\mathbf{r}} = (\tilde{r}^{(1)}, \tilde{r}^{(2)}, \dots, \tilde{r}^{(k-1)})$ ,  $\mathbf{r} = (r^{(1)}, r^{(2)}, \dots, r^{(k-1)}, r^{(k)})$  and  $\lambda_{l-1} = (n_1, \dots, n_{l-1})$ ,  $\lambda_k^* = (n_1^*, \dots, n_k^*)$ . We see that (8.1.14) is a sum over a pair of plane partitions.

## 8.2 Integral representation for $\tau_{\tilde{r},r}$

In case the function  $r$  has zero the scalar product (8.1.1) is degenerate one. For simplicity let us take  $r(0) = 0, r(k) \neq 0, k \leq n$ . The scalar product (8.1.1) is non generate on the subspace of symmetric functions spanned by Schur functions  $\{s_\lambda, l(\lambda) \leq n\}$ ,  $l(\lambda)$  is the length of partition  $\lambda$ .

$$\langle f, g \rangle_{r,n} = \int \cdots \int \prod_{i=1}^n d\bar{z}_i dz_i \mu_r(\bar{z}_i z_i) \Delta_n(\bar{z}) \Delta_n(z) f(\bar{z}^n) g(z^n) \quad (8.2.1)$$

where  $\bar{z}^n = (\bar{z}_1, \dots, \bar{z}_n)$ ,  $z^n = (z_1, \dots, z_n)$  and

$$\Delta_n(\bar{z}) = \prod_{i < j}^n (\bar{z}_i - \bar{z}_j), \quad \Delta_n(z) = \prod_{i < j}^n (z_i - z_j) \quad (8.2.2)$$

Thus get the following integral representation for (8.1.14)

$$\tau_{\tilde{r},r}(\lambda_{l-1}, \lambda_k^*, \gamma, \gamma^*) = \quad (8.2.3)$$

$$\int \cdots \int \prod_{j=1}^{l-1} \prod_{i=1}^{n_j} d\bar{z}_{ji} dz_{ji} \mu_{\tilde{r}(j)}(\bar{z}_{ji} z_{ji}) D_j \delta(z_{ki} - z_{l-1,i}^*) \delta(\bar{z}_{ki} - \bar{z}_{l-1,i}^*) \prod_{j=1}^k \prod_{i=1}^{n_j^*} d\bar{z}_{ji}^* dz_{ji}^* \mu_{r(j)}(\bar{z}_{ji}^* z_{ji}^*) D_j^* \quad (8.2.4)$$

$$D_j = \Delta_{n_j}(\bar{z}_j^*) \Delta_{n_j}(z_j^*) e^{\xi_j}, \quad D_j^* = \Delta_{n_j^*}(\bar{z}_j^*) \Delta_{n_j^*}(z_j^*) e^{\xi_j^*} \quad (8.2.5)$$

$$\xi_j = \sum_{m=1}^{\infty} \sum_{k=1}^{n_j} (z_{jk})^m \left( \gamma_{jm} + \frac{1}{m} \sum_{i=1}^{n_{j-1}} (\bar{z}_{j-1,i})^m \right), j > 1, \quad \xi_1 = \sum_{m=1}^{\infty} \sum_{k=1}^{n_1} (z_{1k})^m \gamma_{1m} \quad (8.2.6)$$

$$\xi_j^* = \sum_{m=1}^{\infty} \sum_{k=1}^{n_j^*} (\bar{z}_{1k}^*)^m \left( \gamma_{jm}^* + \frac{1}{m} \sum_{i=1}^{n_{j-1}^*} (z_{j-1,i}^*)^m \right), j > 1, \quad \xi_1^* = \sum_{m=1}^{\infty} \sum_{k=1}^{n_1^*} (\bar{z}_{1k}^*)^m \gamma_{1m}^* \quad (8.2.7)$$

## 8.3 Multi-matrix integrals

In case  $\tilde{r}^{(j)}(n) = n + a_j, r^{(j)}(n) = n + a_j^*$  we get that the tau function (8.2.3) is equal to the following multi matrix integral

$$\int \cdots \int \prod_{j=1}^{l-1} dM_j d\bar{M}_j e^{Tr M_j \bar{M}_j} e^{V_j} \delta(M_k - M_{l-1}^*) \delta(\bar{M}_{ki} - \bar{M}_{l-1,i}^*) \prod_{j=1}^k dM_j^* d\bar{M}_j^* e^{Tr M_j^* \bar{M}_j^*} e^{V_j^*} \quad (8.3.1)$$

$$V_j = \sum_{m=1}^{\infty} Tr(M_j)^m \left( \gamma_{jm} + \frac{1}{m} Tr(\bar{M}_{j-1})^m \right), j > 1, \quad V_1 = \sum_{m=1}^{\infty} Tr(M_1)^m \gamma_{1m} \quad (8.3.2)$$

$$V_j^* = \sum_{m=1}^{\infty} Tr(\bar{M}_j^*)^m \left( \gamma_{jm}^* + \frac{1}{m} Tr(M_{j-1}^*)^m \right), j > 1, \quad V_1^* = \sum_{m=1}^{\infty} Tr(\bar{M}_1^*)^m \gamma_{1m}^* \quad (8.3.3)$$

where  $M_j, \bar{M}_j$  has a size  $n_j = M + a_j$  and  $M_j^*, \bar{M}_j^*$  has a size  $n_j^* = M + a_j^*$ .

## 8.4 Fermionic representation [9]

Due to (3.3.10) scalar product (8.1.14) is equal to a vacuum expectation value. Let us write down it explicitly using [9].

Formula (5.3.1) is related to 'Gauss decomposition' of operators inside vacuums  $\langle M | \dots | M \rangle$  into diagonal operator  $e^{H_0(\mathbf{T})}$  and upper triangular operator  $e^{H(\mathbf{t})}$  and lower triangular operator

$e^{-H^*(\mathbf{t}^*)}$  the last two have the Toeplitz form. Now let us consider more general two-dimensional Toda chain tau function

$$\tau = \langle M | e^{H(\mathbf{t})} g e^{-A(\mathbf{t}^*)} | M \rangle, \quad (8.4.1)$$

where we decompose  $g$  in the following way:

$$g(\gamma, \gamma^*) = e^{\tilde{A}_1(\gamma_1)} \dots e^{\tilde{A}_k(\gamma_k)} e^{-A_l(\gamma_l^*)} \dots e^{-A_1(\gamma_1^*)}, \quad (8.4.2)$$

where

$$\tilde{\gamma}_i^* = (\gamma_{i1}, \gamma_{i2}, \dots), \quad \gamma_i^* = (\gamma_{i1}^*, \gamma_{i2}^*, \dots) \quad (8.4.3)$$

$$\tilde{A}_k(\gamma_k) = \sum_{m=1}^{\infty} \tilde{A}_{km} \gamma_{km}, \quad A_k(\gamma_k^*) = \sum_{m=1}^{\infty} A_{km} \gamma_{km}^* \quad (8.4.4)$$

Here each of  $A_k(\gamma_k^*)$  has a form as in (3.5.3) and corresponds to operator  $r^{(k)}(D)$ , while each of  $\tilde{A}_k(\tilde{\gamma}_k^*)$  has a form of (3.5.4) and corresponds to operator  $\tilde{r}^{(k)}(D)$ :

$$\tilde{A}_{km} = -\frac{1}{2\pi\sqrt{-1}} \oint \psi^*(z) \left( \tilde{r}^{(k)}(D) z \right)^m \psi(z), \quad m = 1, 2, \dots \quad (8.4.5)$$

$$A_{km} = \frac{1}{2\pi\sqrt{-1}} \oint \psi^*(z) \left( \frac{1}{z} r^{(k)}(D) \right)^m \psi(z), \quad m = 1, 2, \dots, \quad (8.4.6)$$

Collections of variables  $\gamma = \{\gamma_{in}\}, \gamma^* = \{\gamma_{in}^*\}$  play the role of coordinates for some wide enough class of Clifford group elements  $g$ . This tau function is related to rather involved generalization of the hypergeometric functions we considered above. Tau function (8.4.1), (8.4.2) may be considered as the result of applying of the additional symmetries to the vacuum tau function, which is 1, see *Appendix "The vertex operator action"*.

Let us calculate this tau function. First of all we introduce a set consisting of  $m+1$  partitions:

$$(\lambda^1, \dots, \lambda^m, \lambda^{m+1} = \lambda), \quad 0 = \lambda^0 \leq \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m \leq \lambda^{m+1} = \lambda, \quad (8.4.7)$$

see [1] for the notation  $\leq$  for the partitions. The corresponding set

$$\Theta_\lambda^m = (\lambda^1, \theta^1, \dots, \theta^m), \quad \theta^i = \lambda^{i+1} - \lambda^i, \quad i = 1, \dots, m \quad (8.4.8)$$

depends on the partition  $\lambda$  and the number  $m+1$  of the partitions. We take as  $s_{\Theta}(\mathbf{t}^*, \gamma^*)$  the product which is relevant to the set  $\Theta_\lambda^m$  and depending on the set of variables  $\mu_i = \{\mu_{ij}\}$  ( $i = (1, \dots, m+1), j = (1, 2, \dots)$ )

$$s_{\Theta_\lambda^m}(\mu) = s_{\lambda^1}(\mu_1) s_{\theta^1}(\mu_2) \dots s_{\theta^m}(\mu_{m+1}). \quad (8.4.9)$$

Here  $s_{\theta^i}$  is a skew Schur function (see [1]). Further we define function  $r_{\Theta_\lambda^m}(M)$ :

$$r_{\Theta_\lambda^m}(M) = r_{\lambda^1}(M) r_{\theta^1}^1(M) \dots r_{\theta^m}^m(M), \quad (8.4.10)$$

where the function  $r_{\theta^i}^i(M)$ , a skew analogy of  $r_\lambda(M)$  of (2.2.1), is

$$r_{\theta^i}^i(M) = \prod_{j=1}^s r(n_j^{(i)} - j + 1 + M) \dots r(n_j^{(i+1)} - j + M), \quad (8.4.11)$$

where  $\lambda^{i+1} = (n_1^{(i+1)}, \dots, n_s^{(i+1)})$ . If the function  $r^i(m)$  has no poles and zeroes at integer points then the relation

$$r_{\theta^i}^i(M) = \frac{r_{\lambda^{i+1}}^i(M)}{r_{\lambda^i}^i(M)}, \quad i = 1, \dots, m \quad (8.4.12)$$

is correct. To calculate the tau function we need the *Lemma*

**Lemma 3** Let partitions  $\lambda = (i_1, \dots, i_s | j_1 - 1, \dots, j_s - 1)$  and  $\tilde{\mathbf{n}} = (\tilde{i}_1, \dots, \tilde{i}_r | \tilde{j}_1 - 1, \dots, \tilde{j}_r - 1)$  satisfy the relation  $\lambda \geq \tilde{\mathbf{n}}$ . The following is valid:

$$\begin{aligned} \langle 0 | \psi_{i_1}^* \cdots \psi_{i_r}^* \psi_{-\tilde{j}_r} \cdots \psi_{-\tilde{j}_1} e^{A^i(\gamma_i)} \psi_{-j_1}^* \cdots \psi_{-j_s}^* \psi_{i_s} \cdots \psi_{i_1} | 0 \rangle = \\ = (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_\theta(\gamma_i) r_\theta(0), \quad \theta = \lambda - \tilde{\lambda}. \end{aligned} \quad (8.4.13)$$

$$\begin{aligned} \langle 0 | \psi_{i_1}^* \cdots \psi_{i_s}^* \psi_{-j_s} \cdots \psi_{-j_1} e^{\tilde{A}^i(\gamma_i^*)} \psi_{-\tilde{j}_1}^* \cdots \psi_{-\tilde{j}_r}^* \psi_{\tilde{i}_r} \cdots \psi_{\tilde{i}_1} | 0 \rangle = \\ = (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_\theta(\gamma_i) r_\theta(0), \quad \theta = \lambda - \tilde{\lambda}. \end{aligned} \quad (8.4.14)$$

**Proof:** One proof is achieved by a development of  $e^A = 1 + A + \cdots$ ,  $e^{\tilde{A}} = 1 + \tilde{A} + \cdots$  and a direct evaluation of vacuum expectations (8.4.13), (8.4.14). (see Example 22 in Sec 5 of [1] for help).

The second proof of (8.4.13), (8.4.14) is achieved using the developments (3.5.16), (3.5.17) respectively.

Then we obtain the generalization of *Proposition 1*:

### Proposition 15

$$\tau_M(\mathbf{t}, \mathbf{t}^*; \gamma, \gamma^*) = \sum_{\lambda} \sum_{\Theta_{\lambda}^k} \sum_{\Theta_{\lambda}^l} \tilde{r}_{\Theta_{\lambda}^k}(M) r_{\Theta_{\lambda}^l}(M) s_{\Theta_{\lambda}^k}(\mathbf{t}, \gamma) s_{\Theta_{\lambda}^l}(\mathbf{t}^*, \gamma^*), \quad (8.4.15)$$

where  $\tilde{r}_{\Theta_{\lambda}^k}(M)$  and  $r_{\Theta_{\lambda}^l}(M)$  are given by (8.4.11).

With the help of this series one can obtain different hypergeometric functions.

In the end of the subsection we put  $\tilde{\gamma}^* = 0$ . For the case  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$ , all  $\gamma = 0$  one can obtain the analog of (3.4.2)

$$e^{A(\mathbf{t}^*)} g^{-1}(\gamma = 0, \gamma^*) \psi(z) g(\gamma = 0, \gamma^*) e^{-A(\mathbf{t}^*)} = e^{-\xi_{r_l}(\gamma_l^*, z^{-1})} \cdots e^{-\xi_{r_1}(\gamma_1^*, z^{-1})} e^{-\xi_r(\mathbf{t}^*, z^{-1})} \psi(z) \quad (8.4.16)$$

$$e^{A(\mathbf{t}^*)} g^{-1}(\gamma = 0, \gamma^*) \psi^*(z) g(\gamma = 0, \gamma^*) e^{-A(\mathbf{t}^*)} = e^{\xi_{r'_l}(\gamma_l^*, z^{-1})} \cdots e^{\xi_{r'_1}(\gamma_1^*, z^{-1})} e^{\xi_{r'}(\mathbf{t}^*, z^{-1})} \psi^*(z) \quad (8.4.17)$$

where

$$\xi_{r_k}(\mathbf{t}^*, z^{-1}) = \sum_{m=1}^{+\infty} t_m \left( \frac{1}{z} r_k(D) \right)^m, \quad D = z \frac{d}{dz}, \quad r'_k(D) = r_k(-D) \quad (8.4.18)$$

In (8.4.16), (8.4.17)  $\xi_r$  are operators which act on  $z$  variable of  $\psi, \psi^*$ .

In cases  $\mathbf{t} = \mathbf{t}(\mathbf{x}^N)$  (using (5.2.9) and (3.4.3)) and  $\mathbf{t} = -\mathbf{t}(\mathbf{x}^N)$  (using (5.2.10) and (3.4.2)) one gets the following representations

### Proposition 16

$$\tau(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*; \gamma = 0, \gamma^*) = \Delta^{-1} e^{\eta'} \cdots e^{\eta'} e^{\eta'} \Delta, \quad \Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-1-M}} \quad (8.4.19)$$

$$\tau(M, -\mathbf{t}(\mathbf{x}^N), \mathbf{t}^*; \gamma = 0, \gamma^*) = \Delta^{-1} e^{-\eta} \cdots e^{-\eta} e^{-\eta} \Delta, \quad \Delta = \frac{\prod_{i < j} (x_i - x_j)}{(x_1 \cdots x_N)^{N-1+M}} \quad (8.4.20)$$

where

$$\eta = \sum_{i=1}^N \xi_r(\mathbf{t}^*, x_i), \quad \eta_k = \sum_{i=1}^N \xi_{r_k}(\mathbf{t}^*, x_i) \quad (8.4.21)$$

$$\eta' = \sum_{i=1}^N \xi_{r'}(\mathbf{t}^*, x_i), \quad \eta'_k = \sum_{i=1}^N \xi_{r'_k}(\mathbf{t}^*, x_i) \quad (8.4.22)$$

and (cf (3.4.4))

$$\xi_{r'_k}(\mathbf{t}^*, x_i) = \sum_{m=1}^{+\infty} t_m (x_i r_k(D_i))^m, \quad \xi_{r_k}(\mathbf{t}^*, x_i) = \sum_{m=1}^{+\infty} t_m (x_i r_k(-D_i))^m, \quad D_i = x_i \frac{\partial}{\partial x_i} \quad (8.4.23)$$

Let us note that  $[\xi_{r'_k}(\mathbf{t}^*, x_i), \xi_{r'_k}(\mathbf{t}^*, x_j)] = 0$  and  $[\xi_{r_k}(\mathbf{t}^*, x_i), \xi_{r_k}(\mathbf{t}^*, x_j)] = 0$  for all  $k, i, j$ , while in general  $[\eta_k, \eta_n] \neq 0$ .

(8.4.20) may be also obtained from (8.4.19) with the help of (3.4.10).

Looking at (8.4.19) one easily derives a set of linear equations, which are differential equations with respect to variables  $\gamma^*$  and  $\mathbf{t}^*$ :

$$\left( \frac{\partial}{\partial \gamma_{km}^*} - e^{\eta'} \dots e^{\eta_{k-1}'} \cdot \left( \sum_{i=1}^N x_i r_k(D_i) \right)^m \cdot e^{-\eta_{k-1}'} \dots e^{-\eta'} \right) (\Delta \tau(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*; \gamma = 0, \gamma^*)) = 0 \quad (8.4.24)$$

$$\left( \frac{\partial}{\partial t_m^*} - e^{\eta'} \dots e^{\eta'} \cdot \left( \sum_{i=1}^N x_i r_k(D_i) \right)^m \cdot e^{-\eta'} \dots e^{-\eta'} \right) (\Delta \tau(M, \mathbf{t}(\mathbf{x}^N), \mathbf{t}^*; \gamma = 0, \gamma^*)) = 0 \quad (8.4.25)$$

where  $\Delta$  is the same as in (8.4.19).

**Proposition 17** For the tau function  $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$  of (3.4.6) and the vertex operators  $\Omega_{\tilde{r}^1}^{(\infty)}, \Omega_{r^l}^{(0)}$  defined by (5.5.1) we have

$$e^{-\Omega_{\tilde{r}^1}^{(\infty)}(\gamma_1)} \dots e^{-\Omega_{\tilde{r}^k}^{(\infty)}(\gamma_k)} \dots e^{\Omega_{r^l}^{(\infty)}(\gamma_l^*)} \dots e^{\Omega_{r^1}^{(\infty)}(\gamma_1^*)} \cdot \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau(M, \mathbf{t}, \mathbf{t}^*; \gamma, \gamma^*), \quad (8.4.26)$$

$$e^{\Omega_{r^1}^{(0)}(\gamma_1^*)} \dots e^{\Omega_{r^l}^{(0)}(\gamma_l^*)} \dots e^{-\Omega_{\tilde{r}^k}^{(0)}(\gamma_k)} \dots e^{-\Omega_{\tilde{r}^1}^{(0)}(\gamma_1)} \cdot \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau(M, \mathbf{t}, \mathbf{t}^*; \gamma, \gamma^*). \quad (8.4.27)$$

## 8.5 The example of Gelfand, Graev and Retakh hypergeometric series [9]

Below we also put  $\gamma = 0$ . Let us consider the tau function:

$$\tau(M, \tilde{\beta}, \beta; 0, \gamma^*) = \langle M | e^{\tilde{A}(\tilde{\beta})} e^{-A_l(\gamma_l^*)} \dots e^{-A_1(\gamma_1^*)} e^{-A(\beta)} | M \rangle. \quad (8.5.1)$$

We put

$$\tilde{\beta} = (x, \frac{x^2}{2}, \frac{x^3}{3}, \dots), \quad \beta = (y_1, 0, 0, \dots), \quad \gamma_i^* = (y_{i+1}, 0, 0, \dots) \quad i = (1, \dots, l). \quad (8.5.2)$$

We obtain the series

$$\tau(M, x, y_1, \dots, y_{l+1}) = \sum_{n_1, \dots, n_{l+1}=0}^{+\infty} \tilde{r}_{(n_1+\dots+n_{l+1})}(M) r_{\Theta_\lambda^l}(M) \frac{(xy_1)^{n_1} \dots (xy_{l+1})^{n_{l+1}}}{n_1! \dots n_{l+1}!} = \quad (8.5.3)$$

$$\sum_{n_1, \dots, n_{l+1} \in \mathbb{Z}} c(n_1, \dots, n_{l+1}) (xy_1)^{n_1} \dots (xy_{l+1})^{n_{l+1}}, \quad c(n_1, \dots, n_{l+1}) = \frac{\tilde{r}_{(n_1+\dots+n_{l+1})}(M) r_{\Theta_\lambda^l}(M)}{\Gamma(n_1+1) \dots \Gamma(n_{l+1}+1)}, \quad (8.5.4)$$

where  $\Theta_\lambda^l$  corresponds to the set of simple partitions-rows

$$\lambda^1 = (n_1), \lambda^2 = (n_1 + n_2), \dots, \lambda_{l+1} = (n_1 + \dots + n_{l+1}) \quad (8.5.5)$$

When functions  $b_i(n_1, \dots, n_{l+1})$  defined as

$$b_i(n_1, \dots, n_{l+1}) = \frac{c(n_1, \dots, n_i + 1, \dots, n_{l+1})}{c(n_1, \dots, n_{l+1})}, \quad i = 1, \dots, l+1 \quad (8.5.6)$$

are rational functions of  $(n_1, \dots, n_{l+1})$ , then tau function (8.5.3) is a Horn hypergeometric series [21].

Above series for the special choice of functions  $r^i(D)$  can be deduced from the Gelfand, Graev and Retakh series [46] defined on the special lattice and corresponding to the special set of parameters. Let us take the rational functions  $r^i(D)$ :

$$r^i(D) = \frac{\prod_{j=1}^{p(i)} (D + a_j^{(i)})}{\prod_{m=1}^{s(i)} (D + b_m^{(i)})}, \quad (i = 0, \dots, l), \quad r^0(D) = r(D) \quad (8.5.7)$$

$$\tilde{r}(D) = \frac{\prod_{j=1}^{p^{(l+1)}} (D + a_j^{(l+1)})}{\prod_{m=1}^{s^{(l+1)}} (D + b_m^{(l+1)})} \quad (8.5.8)$$

Let define  $N = p^{(0)} + s^{(0)} + 2 \sum_{j=1}^l (p^{(j)} + 2s^{(j)}) + p^{(l+1)} + s^{(l+1)} + l + 1$  and consider complex space  $C^N$ . In this space we consider the  $l + 1$ -dimensional basis  $B$  and the vector  $v$  consisting of parameters.

$$\begin{aligned} p_0 = s_0 = 0, \quad p_i = p^{(i-1)} + s^{(i)}, \quad s_i = s^{(i-1)} + p^{(i)}, \quad i = (1, \dots, l) \\ p_{l+1} = p^{(l)} + p^{(l+1)}, \quad s_{l+1} = s^{(l)} + s^{(l+1)}, \quad N = \sum_{j=1}^{l+1} (p_j + s_j) + l + 1 \end{aligned} \quad (8.5.9)$$

$$\begin{aligned} \mathbf{f}^i = -(\mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+1} + \dots + \mathbf{e}_{p_1+s_1+\dots+p_{i-1}+s_{i-1}+p_i}) + \\ + (\mathbf{e}_{p_1+s_1+\dots+p_{i-1}+s_{i-1}+p_i+1} + \dots + \mathbf{e}_{p_1+s_1+\dots+p_{i-1}+s_{i-1}+p_i+s_i}), \quad i = 1, \dots, l+1 \end{aligned} \quad (8.5.10)$$

where  $\mathbf{e}_i = \underbrace{(0, \dots, 0, \hat{1}, 0, \dots)}_N$ . The lattice  $B \in C^N$  is generated by the vector basis of dimension  $l + 1$ :

$$\mathbf{b}^i = \mathbf{f}^i + \dots + \mathbf{f}^{l+1} + \mathbf{e}_{N-l-1+i}, \quad i = 1, \dots, l+1 \quad (8.5.11)$$

Vector  $v \in C^N$  is defined as follows (compare with (8.5.10)):

$$\begin{aligned} v^i = -(a_1^{(i-1)} \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+1} + \dots + a_{p^{(i-1)}}^{(i-1)} \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p^{(i-1)}} + \\ + b_1^{(i)} \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p^{(i-1)}+1} + \dots + b_{s^{(i)}}^{(i)} \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i}) + \\ + ((b_1^{(i-1)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+1} + \dots + (b_{s^{(i-1)}}^{(i-1)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}} + \\ + (a_1^{(i)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}+1} + \dots + (a_{s^{(i)}}^{(i)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s_i}) \end{aligned} \quad (8.5.12)$$

for  $i = (1, \dots, l)$ , and

$$\begin{aligned} v^{l+1} = -(a_1^{(l)} \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+1} + \dots + a_{p^{(l)}}^{(l)} \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p^{(l)}} + \\ + a_1^{(l+1)} \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p^{(l)}+1} + \dots + a_{s^{(l+1)}}^{(l+1)} \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p_{l+1}}) + \\ + ((b_1^{(l)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p_{l+1}+1} + \dots + (b_{s^{(l)}}^{(l)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p_{l+1}+s^{(l)}} + \\ + (b_1^{(l+1)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p_{l+1}+s^{(l)}+1} + \dots + (b_{s^{(l+1)}}^{(l+1)} - 1) \mathbf{e}_{p_0+s_0+\dots+p_l+s_l+p_{l+1}+s_{l+1}}) \end{aligned} \quad (8.5.13)$$

Vector  $v$  is:

$$v = v^1 + \dots + v^{l+1} \quad (8.5.14)$$

Now we can write down Gelfand, Graev and Retakh hypergeometric series corresponding to the lattice  $B$  and vector  $v$ :

$$F_B(v; z) = \sum_{\mathbf{b} \in B} \prod_{j=1}^N \frac{z_j^{v_j+b_j}}{\Gamma(v_j+b_j+1)} + \sum_{n_1, \dots, n_{l+1} \in \mathbb{Z}} \prod_{j=1}^N \frac{z_j^{v_j+n_1 b_j^1 + \dots + n_{l+1} b_j^{l+1}}}{\Gamma(v_j+n_1 b_j^1 + \dots + n_{l+1} b_j^{l+1} + 1)} \quad (8.5.15)$$

Let us compare this series with tau function (8.5.3):

$$F_B(v; \mathbf{z}) = c_1(a, b) g_1(\mathbf{z}) \cdots c_{l+1}(a, b) g_{l+1}(\mathbf{z}) \tau(M, x, y_1, \dots, y_{l+1}) \quad (8.5.16)$$

where

$$c_i^{-1}(a, b) = \Gamma(1 - a_1^{(i-1)}) \cdots \Gamma(1 - a_{p^{(i-1)}}^{(i-1)}) \Gamma(1 - b_1^{(i)}) \cdots \Gamma(1 - b_{s^{(i)}}^{(i)}) \times \\ \times \Gamma(b_1^{(i-1)}) \cdots \Gamma(b_{s^{(i-1)}}^{(i-1)}) \Gamma(a_1^{(i)}) \cdots \Gamma(a_{s^{(i)}}^{(i)}), \quad i = 1, \dots, l \quad (8.5.17)$$

$$c_{l+1}^{-1}(a, b) = \Gamma(1 - a_1^{(l)}) \cdots \Gamma(1 - a_{p^{(l)}}^{(l)}) \Gamma(1 - a_1^{(l+1)}) \cdots \Gamma(1 - a_{s^{(l+1)}}^{(l+1)}) \times \\ \times \Gamma(b_1^{(l)}) \cdots \Gamma(b_{s^{(l)}}^{(l)}) \Gamma(b_1^{(l+1)}) \cdots \Gamma(b_{s^{(l+1)}}^{(l+1)}) \quad (8.5.18)$$

$$\frac{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+1} \cdots z_{p_1+s_1+\dots+p_{i-1}+s_{i-1}+p_i+s_i}}{(-z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+1}) \cdots (-z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i})} = 1, \quad i = 2, \dots, l \quad (8.5.19)$$

$$\frac{z_{p_1+1} \cdots z_{p_1+s_1} z_{N-l}}{(-z_1) \cdots (-z_{p_1})} = y_1 \quad (8.5.20)$$

$$y_i = z_{N-l-1+i}, \quad i = 2, \dots, l+1 \quad (8.5.21)$$

$$\frac{z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+1} \cdots z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+s_{l+1}}}{(-z_{p_1+s_1+\dots+p_l+s_l+1}) \cdots (-z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}})} = x \quad (8.5.22)$$

$$g_i(\mathbf{z}) = \frac{\left(-a_1^{(i-1)}\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+1}} \cdots \frac{\left(-a_{p^{(i-1)}}^{(i-1)}\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p^{(i-1)}}} \times \\ \times \frac{\left(-b_1^{(i)}\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p^{(i-1)}+1}} \cdots \frac{\left(-b_{s^{(i)}}^{(i)}\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i}} \times \\ \times \frac{\left(b_1^{(i-1)}-1\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+1}} \cdots \frac{\left(b_{s^{(i-1)}}^{(i-1)}-1\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}}} \times \\ \times \frac{\left(a_1^{(i)}-1\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}+1}} \cdots \frac{\left(a_{s^{(i)}}^{(i)}-1\right)}{z_{p_0+s_0+\dots+p_{i-1}+s_{i-1}+p_i+s_i-1}} \quad (8.5.23)$$



for  $i = (1, \dots, l)$ , and

$$\begin{aligned}
g_{l+1}(\mathbf{z}) = & z_{p_1+s_1+\dots+p_l+s_l+1}^{\binom{-a_1^{(l)}}{p^{(l)}}} \cdots z_{p_1+s_1+\dots+p_l+s_l+p^{(l)}}^{\binom{-a_p^{(l)}}{p^{(l)}}} \times \\
& \times z_{p_1+s_1+\dots+p_l+s_l+p^{(l)}+1}^{\binom{-a_1^{(l+1)}}{p^{(l+1)}}} \cdots z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}}^{\binom{-a_{s^{(l+1)}}^{(l+1)}}{p^{(l+1)}}} \times \\
& \times z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+1}^{\binom{b_1^{(l)}-1}{p^{(l)}}} \cdots z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+s^{(l)}}^{\binom{b_{s^{(l)}}^{(l)}-1}{p^{(l)}}} \times \\
& \times z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+s^{(l)}+1}^{\binom{b_1^{(l+1)}-1}{p^{(l+1)}}} \cdots z_{p_1+s_1+\dots+p_l+s_l+p_{l+1}+s_{l+1}-1}^{\binom{b_{s^{(l+1)}}^{(l+1)}-1}{p^{(l+1)}}}
\end{aligned} \tag{8.5.24}$$

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